Advanced Quantum Mechanics

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Harmonic Oscillator operators

Derivations of Harmonic oscillator operators First of all, lets derive $\hat{H}$, $\hat{x}$, $\hat{x}^2$, $\hat{p}$, $\hat{p}^2$ operators for harmonic oscillator. Lets write Schrodinger equation for one-dimensional harmonic oscillator.

$$\left[\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2\right] = E_n\psi_n(x)$$

also in Dirac notation:

$$\hat{H}n = E_n n$$

as we see Hamiltonian is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

lets rewrite Hamiltonian as:

$$\hat{H} = \omega \left[\hat{x}\sqrt{\frac{m\omega}{2}} - i\hat{p}\frac{1}{\sqrt{2m\omega}} \right] \left[\hat{x}\sqrt{\frac{m\omega}{2}} + i\hat{p}\frac{1}{\sqrt{2m\omega}} \right]$$

then we have:

$$\omega\hbar \left[\hat{x}\sqrt{\frac{m\omega}{2\hbar}} - i\hat{p}\frac{1}{\sqrt{2m\hbar\omega}} \right] \left[\hat{x}\sqrt{\frac{m\omega}{2\hbar}} + i\hat{p}\frac{1}{\sqrt{2m\hbar\omega}} \right] = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{i\omega}{2}[x, p]$$

Lets introduce the notation:

$$\boxed{\hat{a}_- = \hat{x}\sqrt{\frac{m\omega}{2\hbar}} + i\hat{p}\frac{1}{\sqrt{2m\hbar\omega}}}$$

$$\boxed{\hat{a}_+ = \hat{x}\sqrt{\frac{m\omega}{2\hbar}} - i\hat{p}\frac{1}{\sqrt{2m\hbar\omega}}}$$

Those are the ladder operators and we can therefore write Hamiltonian as :

$$\boxed{\hat{H} = \hbar\omega\Big(\hat{a}_+\hat{a}_- + \frac{1}{2}\Big)}$$

Lets find:

$$[\hat{H}, \hat{a}_-] = [\hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2}), \hat{a}_-] = \hbar\omega[\hat{a}_+\hat{a}_-, \hat{a}_-] \quad since \quad [\frac{1}{2}, \hat{a}_-] = 0$$

Now:

$$[\hat{a}_+\hat{a}_-, \hat{a}_-] = \hat{a}_+\hat{a}_-, \hat{a}_- - \hat{a}_+\hat{a}_-, \hat{a}_- = [\hat{a}_+\hat{a}_-]\hat{a}_- = -\hat{a}_-$$

$$since[\hat{a}_-, \hat{a}_+] = 1 \quad and \quad [\hat{a}_+, \hat{a}_-] = -1$$

Using this relations:

$$[\hat{H}, \hat{a}_-] = -\hbar\omega\hat{a}_-$$

Using same analogy we can define that:

$$[\hat{H}, \hat{a}_+] = \hbar\omega\hat{a}_+$$

Let us now compute:

$$\hat{H}\Big(\hat{a}_- n\Big) = \hat{a}_-\hat{H}n + [\hat{H}, \hat{a}_-]n = E_n\hat{a}_- n - \hbar\omega\hat{a}_- n = (E_n - \hbar\omega)\Big(\hat{a}_- n\Big)$$

Using same analogy:

$$\hat{H}\hat{a}_+ n = (E_n + \hbar\omega)\hat{a}_+ n$$

We can summarize these results:

$$\hat{a}_+ n = c_n n + 1$$

$$\hat{a}_- n = d_n n - 1$$

Where c_n and d_n are constants of proportionality (NOT eigenvalues!) and,

$$\hat{H} n + 1 = E_{n+1} n + 1 = (E_n + \hbar\omega) n + 1$$

$$\hat{H} n - 1 = E_{n-1} n - 1 = (E_n - \hbar\omega) n - 1$$

Since $\hat{a}_- 0 = 0$ Lets find:

$$\hat{H}0 = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})0 = \frac{1}{2}\hbar\omega 0 = E_0 0$$

In general form:

$$\boxed{\hat{H}n = E_n n = (n + \frac{1}{2})\hbar\omega n \quad n = 0, 1, 2, 3, ...}$$