Advanced Quantum Mechanics

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Harmonic Oscilator part2: Let's normalize the Eigenstates to find the constants of c_n and d_n:

$$n\hat{a}_+\hat{a}_-n = c_n n\hat{a}_+ n - 1 = c_n n - 1\hat{a}_- n^* =$$

$$= c_n c_n^* n - 1 n - 1 = |c_n|^2$$

Since we know that $\hat{a}_+\hat{a}_- = \frac{\bar{H}}{\hbar\omega} - \frac{1}{2}$ then:

$$n\hat{a}_+\hat{a}_-n = nbranketn = nquadc_n = \sqrt{n}$$

A similar calculation shows that:

$$n\hat{a}_-\hat{a}_+n = |d_n|^2 = n + 1 \quad d_n = \sqrt{n+1}$$

In summary:

$$\boxed{\hat{a}_+ n = \sqrt{n+1} n+1}$$

$$\boxed{\hat{a}_- n = \sqrt{n} n-1}$$

Now lets derive position and momentum operators by adding and subtracting the expressions of $\hat{a}_-$ and $\hat{a}_+$:

$$\hat{a}_- + \hat{a}_+ = 2\sqrt{\frac{m\omega}{2\hbar}}\hat{x} \quad and \quad \hat{a}_- - \hat{a}_+ = 2\sqrt{\frac{m\omega}{2\hbar}}\frac{1}{m\omega}i\hat{p}$$

then:

$$\boxed{\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left[\hat{a}_+ + \hat{a}_-\right]}$$

$$\boxed{\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left[\hat{a}_+ - \hat{a}_-\right]}$$

In this part we are trying to find expected values of $\hat{H}$, $\hat{x}$, $\hat{x}^2$, $\hat{p}$, $\hat{p}^2$ for state vector of one dimensional harmonic oscillator.

$$\Psi = \frac{1}{\sqrt{6}} \begin{vmatrix} 1 \\ 2 \\ 1 \\ 0 \\ . \\ . \\ . \end{vmatrix}$$

Find $<\hat{H}>$:

$$<\hat{H}>=\int \Psi^{*}\hat{H}\Psi dx=\Psi\hat{H}\Psi=$$

$$=\frac{1}{6}\Big[1+2+1+0+...\Big]\hat{H}\Big[1+2+1+0+...\Big]=$$

Hamiltonian operators acts on state gives the energy of same state as eigenvalues:

$$=\frac{1}{6}\Big[1+2+1+0+...\Big]\Big[E_1 1+E_2 2+E_1 1+E_0 0+...\Big]=$$

$$=\frac{1}{6}\Big[1E_1 1+2E_2 2+1E_1 1+0E_0 0+...\Big]=$$

$$=\frac{1}{6}\Big[E_1+E_2+E_1+E_0+...\Big]$$

Lets use $E_n n = (n + \frac{1}{2})\hbar\omega n$ to find,
$E_1 = \frac{3}{2}\hbar\omega$,$E_0 = \frac{1}{2}\hbar\omega$,$E_2 = \frac{5}{2}\hbar\omega$ then:

$$< \hat{H} >= \frac{1}{6}\big[\frac{3}{2}\hbar\omega + \frac{5}{2}\hbar\omega + \frac{3}{2}\hbar\omega + \frac{1}{2}\hbar\omega ...\Big] = \frac{1}{6}\Big[6\hbar\omega + ...\Big]$$

Part b Find $<\hat{x}>$, $<\hat{x}^2>$, $<\hat{p}>$ and $<\hat{p}^2>$. We derived that $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left[\hat{a}_+ + \hat{a}_-\right]$

$$<\hat{x}> = \int \Psi^* \hat{x} \Psi dx = \Psi \hat{x} \Psi =$$

$$= \frac{1}{6}\left[1+2+1+0+...\right]\sqrt{\frac{\hbar}{2m\omega}}\left[\hat{a}_+ + \hat{a}_-\right]\left[1+2+1+0+...\right] =$$

$$= \frac{1}{6}\sqrt{\frac{\hbar}{2m\omega}}\left[1+2+1+0+...\right]\left[\hat{a}_+1+\hat{a}_+2+\hat{a}_+1+\hat{a}_+0+\hat{a}_-1+\hat{a}_-2+\hat{a}_-1+\hat{a}_-0\right.$$

using the relation $\boxed{\hat{a}_+ n = \sqrt{n+1} n+1}$ and $\boxed{\hat{a}_- n = \sqrt{n} n-1}$:

$$= \frac{1}{6}\sqrt{\frac{\hbar}{2m\omega}}\left[1+2+1+0+...\right]\left[\sqrt{2}2+\sqrt{3}3+\sqrt{2}2+1+0+\sqrt{2}1+0+0+...\right] =$$

$$\frac{1}{6}\sqrt{\frac{\hbar}{2m\omega}}\left[2(1+\sqrt{2})11+2\sqrt{2}22+200...\right] = \frac{4+4\sqrt{2}}{6}\sqrt{\frac{\hbar}{2m\omega}} = \frac{2+2\sqrt{2}}{3}\sqrt{\frac{\hbar}{2m\omega}}$$

$$\boxed{<\hat{x}> = \int \Psi^* \hat{x} \Psi dx = \Psi \hat{x} \Psi = \frac{2+2\sqrt{2}}{3}\sqrt{\frac{\hbar}{2m\omega}}}$$

First lets find what is $\hat{x}^2$:

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\Big[\hat{a}_+ + \hat{a}_-\Big]^2 = \frac{\hbar}{2m\omega}\Big[\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2\Big]$$

Now we can write:

$$< \hat{x}^2 >= \int \Psi^* \hat{x}^2 \Psi dx = \Psi \hat{x}^2 \Psi =$$

$$= \frac{1}{6}\frac{\hbar}{2m\omega}\left[1+2+1+0+...\right]\left[\hat{a}_+^2+\hat{a}_+\hat{a}_-+\hat{a}_-\hat{a}_++\hat{a}_-^2\right]\left[1+2+1+0+...\right]$$