Advanced Quantum Mechanics

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Harmonic Oscillator part3: Lets solve:

$$\left[\hat{a}_+^2+\hat{a}_+\hat{a}_-+\hat{a}_-\hat{a}_++\hat{a}_-^2\right]\left[1+2+1+0+...\right] = \left(\hat{a}_+^2 1+\hat{a}_+^2 2+\hat{a}_+^2 1+\hat{a}_+^2 0\right)+$$

$$+\left(\hat{a}_+\hat{a}_- 1+\hat{a}_+\hat{a}_- 2+\hat{a}_+\hat{a}_- 1+\hat{a}_+\hat{a}_- 0\right)+\left(\hat{a}_-\hat{a}_+ 1+\hat{a}_-\hat{a}_+ 2+\hat{a}_-\hat{a}_+ 1+\hat{a}_-\hat{a}_+ 0\right)$$

$$+\left(\hat{a}_-^2 1 + \hat{a}_-^2 2 + \hat{a}_-^2 1 + \hat{a}_-^2 0\right) =$$

$$= \left(\sqrt{6}3 + \sqrt{12}4 + \sqrt{6}3 + \sqrt{2}2\right) + \left(1 + \sqrt{4}2 + 1 + 0 + \right) +$$

$$+\left(\sqrt{4}1 + \sqrt{9}2 + \sqrt{4}1 + 0\right) + \left(0 + \sqrt{2}0 + 0 + 0\right) =$$

$$= 2\sqrt{6}3 + \sqrt{12}4 + (\sqrt{2} + \sqrt{4} + \sqrt{9})2 + (2\sqrt{4} + 2)1 + (1 + \sqrt{2})0$$

$$= 2\sqrt{6}3 + 2\sqrt{3}4 + (\sqrt{2} + 5)2 + 61 + (1 + \sqrt{2})0$$

Finally our equation is:

$$= \frac{1}{6}\frac{\hbar}{2m\omega}\left[1{+}2{+}1{+}0{+}...\right]\left[2\sqrt{6}3{+}2\sqrt{3}4{+}(\sqrt{2}{+}5)2{+}61{+}(1{+}\sqrt{2})0\right] =$$

$$= \frac{1}{6}\frac{\hbar}{2m\omega}\left[(611 + (\sqrt{2} + 5)22 + 611 + (1 + \sqrt{2})00\right] =$$

$$= \frac{1}{6}\frac{\hbar}{2m\omega}\left[18 + 2(\sqrt{2})\right] = \frac{1}{6}\frac{\hbar}{m\omega}\left[9 + (\sqrt{2})\right]$$

$$\boxed{< \hat{x^2} >= \int \Psi^* \hat{x^2} \Psi dx = \Psi \hat{x^2} \Psi = \frac{1}{6}\frac{\hbar}{m\omega}\left[9 + (\sqrt{2})\right]}$$

Now, lets find σ_x:

$$\boxed{< \hat{x} >= \frac{2+2\sqrt{2}}{3}\sqrt{\frac{\hbar}{2m\omega}}} \quad and \quad \boxed{< \hat{x}^2 >= \frac{1}{6}\frac{\hbar}{m\omega}\left[9+(\sqrt{2})\right]}$$

$$\sigma_x = \sqrt{< \hat{x}^2 > - < \hat{x} >^2} = \sqrt{\frac{1}{6}\frac{\hbar}{m\omega}\left[9+(\sqrt{2})\right] - \frac{12+8\sqrt{2}}{9}\frac{\hbar}{2m\omega}}$$

$$\boxed{\sigma_x = \sqrt{< \hat{x}^2 > - < \hat{x} >^2} = \sqrt{\frac{(15-5\sqrt{2})\hbar}{18m\omega}}}$$

We found that $\boxed{\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\Big[\hat{a}_{+} - \hat{a}_{-}\Big]}$ lets use it:

$$< \hat{x} >= \int \Psi^{*}\hat{x}\Psi dx = \Psi\hat{x}\Psi =$$

$$= \frac{1}{6}\Big[1 + 2 + 1 + 0 + ...\Big] i\sqrt{\frac{m\omega\hbar}{2}}\Big[\hat{a}_{+} - \hat{a}_{-}\Big]\Big[1 + 2 + 1 + 0 + ...\Big] =$$

$$= \frac{1}{6} i\sqrt{\frac{m\omega\hbar}{2}}\Big[1+2+1+0+...\Big]\Big[\hat{a}_{+}1+\hat{a}_{+}2+\hat{a}_{+}1+\hat{a}_{+}0-\hat{a}_{-}1-\hat{a}_{-}2-\hat{a}_{-}1-\hat{a}_{-}0$$

using the relation $\boxed{\hat{a}_+ n = \sqrt{n+1} n + 1}$ and $\boxed{\hat{a}_- n = \sqrt{n} n - 1}$:

$$= \frac{1}{6} i \sqrt{\frac{m\omega\hbar}{2}} \Big[1 + 2 + 1 + 0 \Big] \Big[\sqrt{2} 2 + \sqrt{3} 3 + \sqrt{2} 2 + 1 - 0 - \sqrt{2} 1 - 0 - 0 \Big] =$$

$$\frac{1}{6} i \sqrt{\frac{m\omega\hbar}{2}} \Big[2(1 - \sqrt{2}) 11 + 2\sqrt{2} 22 - 200 \Big] = 0$$

$$\boxed{< \hat{p} >} = 0$$

Find $< \hat{p}^2 >$ First lets find what is $\hat{p}^2$:

$$\hat{p}^2 = (i\sqrt{\frac{m\omega\hbar}{2}}\left[\hat{a}_+ - \hat{a}_-\right])^2 = -\frac{m\omega\hbar}{2}\left[\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right]$$

Now we can write:

$$< \hat{p}^2 >= \int \Psi^* \hat{p}^2 \Psi dx = \Psi \hat{x}^2 \Psi =$$

$$= -\frac{1}{6}\frac{m\omega\hbar}{2}\Big[1+2+1+0+...\Big]\Big[\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2\Big]\Big[1+2+1+0+...\Big]$$

Lets solve:

$$\left[\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right]\left[1+2+1+0+...\right] = \left(\hat{a}_+^2 1 + \hat{a}_+^2 2 + \hat{a}_+^2 1 + \hat{a}_+^2 0\right) -$$

$$-\left(\hat{a}_+\hat{a}_- 1 + \hat{a}_+\hat{a}_- 2 + \hat{a}_+\hat{a}_- 1 + \hat{a}_+\hat{a}_- 0\right) - \left(\hat{a}_-\hat{a}_+ 1 + \hat{a}_-\hat{a}_+ 2 + \hat{a}_-\hat{a}_+ 1 + \hat{a}_-\hat{a}_+ 0\right)$$

$$+\left(\hat{a}_-^2 1 + \hat{a}_-^2 2 + \hat{a}_-^2 1 + \hat{a}_-^2 0\right) =$$

$$= \left(\sqrt{6}3 + \sqrt{12}4 + \sqrt{6}3 + \sqrt{2}2\right) - \left(1 + \sqrt{4}2 + 1 + 0 + \right) -$$

$$-\left(\sqrt{4}1 + \sqrt{9}2 + \sqrt{4}1 + 0\right) + \left(0 + \sqrt{2}0 + 0 + 0\right) =$$

$$= 2\sqrt{6}3 + \sqrt{12}4 + (\sqrt{2} - \sqrt{4} - \sqrt{9})2 + (-2 - 2\sqrt{4})1 + (-1 + \sqrt{2})0 =$$

$$= 2\sqrt{6}3 + 2\sqrt{3}4 + (\sqrt{2} - 5)2 - 61(\sqrt{2} - 1)0$$

Finally our equation is:

$$= -\frac{1}{6}\frac{m\omega\hbar}{2}\Big[1+2+1+0+...\Big]\Big[2\sqrt{6}3+2\sqrt{3}4+(\sqrt{2}-5)2-61+(\sqrt{2}-1)0\Big] =$$

$$= -\frac{1}{6}\frac{m\omega\hbar}{2}\Big[-611+(\sqrt{2}-5)22-611+(\sqrt{2}-1)00\Big] = -\frac{m(-18+2\sqrt{2})\omega\hbar}{12}$$

$$\boxed{< \hat{p}^2 >= \frac{(9-\sqrt{2})m\omega\hbar}{6}}$$

lets find σ_p

$$\sigma_p = \sqrt{<\hat{p}^2> - <\hat{p}>^2} = \sqrt{\frac{(9-\sqrt{2})m\omega\hbar}{6}}$$

Lets check the uncertainty principle $\sigma_x\sigma_p \geq \frac{\hbar}{2}$

$$\sigma_x\sigma_p = \sqrt{\frac{(15-5\sqrt{2})\hbar}{18m\omega}}\sqrt{\frac{(9-\sqrt{2})m\omega\hbar}{6}}$$