Advanced Quantum Mechanics

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Angular momentum

We need to find eigenvectors and eigenvalues of angular momentum operators on system which angular momentum $l = 1$. Lets quickly derive and write down some important equations. Classical angular momentum vector is denoted $\vec{L} = \vec{r} \times \vec{p}$:

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - yp_x)\hat{k} =$$

$$= L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

we might write L in this form:

$$L = \begin{vmatrix} -i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y} \\ -i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z} \\ -i\hbar x\frac{\partial}{\partial y} + i\hbar y\frac{\partial}{\partial x} \end{vmatrix}$$

Also lets not forget $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$ and $[L_z, L_x] = i\hbar L_y$. Lets find ladder operators using relation

$$L^2 = L_x^2 + L_y^2 + L_z^2 \quad L^2 - L_z^2 = L_x^2 + L_y^2$$

The sum of the components $L_x^2 + L_y^2$ would appear to factor $(L_x + iL_y)(L_x - iL_y)$.
We will use the notations $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$ to define the operators. Those operators are not Hermitian.
Lets derive useful commutation relation.

$$[L_z, L\pm] = i\hbar L_y \pm \hbar L_x = \pm\hbar L_\pm$$

as well as:

$$[L_+, L_-] = -i[L_x, L_y] + i[L_y, L_x] = 2\hbar L_z$$

Suppose we have state ψ. Let the eigenvalues be ν and λ respectively.

$$L_z\psi = \nu\psi \quad \vec{L}^2\psi = \lambda\psi$$

Now consider $L_+\psi$ and $L_-\psi$. Acting with L_z on these states we find:

$$L_zL_+\psi = L_+L_z\psi + [L_z, L_+]\psi = \nu L_+\psi + \hbar L_+\psi = (\nu + \hbar)L_+\psi$$

$$L_zL_-\psi = L_-L_z\psi + [L_z, L_-]\psi = \nu L_-\psi - \hbar L_-\psi = (\nu - \hbar)L_-\psi$$

If we act $\vec{L}^2$ on these states we get

$$\vec{L}^2 L_+\psi = L_+\vec{L}^2\psi + [\vec{L}^2, L_+]\psi = \nu L_+\psi + 0 = \lambda L_+\psi$$

$$\vec{L}^2 L_-\psi = L_-\vec{L}^2\psi + [\vec{L}^2, L_-]\psi = \nu L_-\psi + 0 = \lambda L_-\psi$$

Now lets write eigen values of some operators:

$$\boxed{L^2 l, m = l(l+1)\hbar^2 l, m}$$

$$\boxed{L_z l, m = m\hbar l, m}$$

$$\boxed{L_+ l, m = \hbar\sqrt{l(l+1) - m(m+1)} l, m+1} \qquad \boxed{L_+ l, m_{max} = 0}$$

$$\boxed{L_- l, m = \hbar\sqrt{l(l+1) - m(m-1)} l, m-1} \qquad \boxed{L_- l, m_{min} = 0}$$

$$\boxed{L_x = \frac{L_+ + L_-}{2}}$$

$$\boxed{L_y = \frac{L_+ - L_-}{2i}}$$

We can rewrite

$$\hat{L}_x\hat{L}_y+\hat{L}_y\hat{L}_x = \hat{L}_x\hat{L}_y-\hat{L}_y\hat{L}_x+2\hat{L}_y\hat{L}_x = [\hat{L}_x, \hat{L}_y]+2\hat{L}_y\hat{L}_x = i\hbar L_z+2\hat{L}_y\hat{L}_x$$

For $l = 1$ case the dimension is $2l + 1 = 3$ ad with,

$$1,1 = \begin{vmatrix}1\\0\\0\end{vmatrix} \quad 1,0 = \begin{vmatrix}0\\1\\0\end{vmatrix} \quad 1,-1 = \begin{vmatrix}0\\0\\1\end{vmatrix}$$

Lets find L_z matrix:

$$L_z 1,1 = 1\hbar \begin{vmatrix}1\\0\\0\end{vmatrix} \quad L_z 1,0 = 0 \begin{vmatrix}0\\1\\0\end{vmatrix} \quad L_z 1,-1 = -1 \begin{vmatrix}0\\0\\1\end{vmatrix}$$

then we can rewrite that

$$\boxed{\hat{L}_z = \hbar \begin{vmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{vmatrix}}$$

Lets find L_+ matrix:

$$L_+1,1 = 0 \quad L_+1,0 = \hbar\sqrt{2}\begin{vmatrix}1\\0\\0\end{vmatrix} \quad L_+1,-1 = \hbar\sqrt{2}\begin{vmatrix}0\\1\\0\end{vmatrix}$$

Lets be more clear lets assume that :

$$L_+ = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_2\end{vmatrix}$$

if:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} = \hbar\sqrt{2} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \hbar\sqrt{2} \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$$