Sum of Independent Random Variables – Convolution

Consider two independent random variables X and Y. Let

$$Z = X + Y$$

For the **discrete** case: $p_X(x)$ and $p_Y(y)$

For any integer z,

$$\begin{aligned} p_Z(z) &= P(X + Y = z) \\ &= \sum_{\{(x,y)\mid x+y=z\}} P(X = x, Y = y) \\ &= \sum_x P(X = x, Y = z - x) \\ &= \sum_x p(x, z - x) \end{aligned}$$

By the definition of independence

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z - x)$$

Note that p_Z is the **"convolution"** of $p_X(x)$ and $p_Y(y)$.

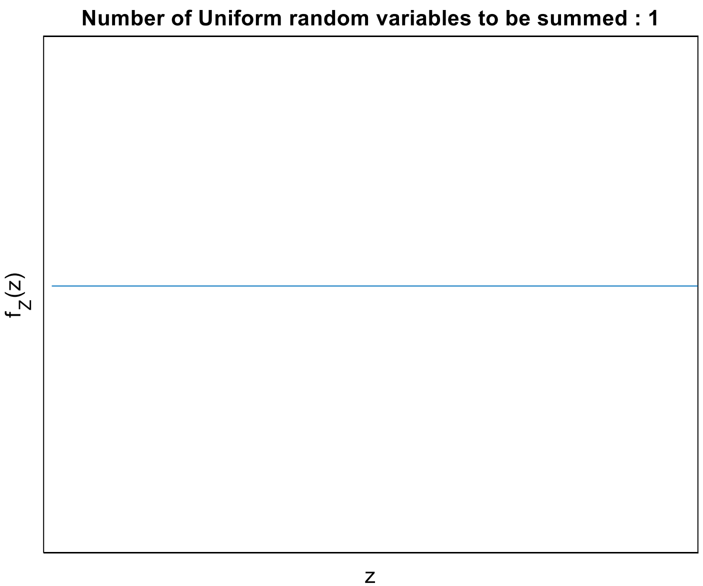
Textbook: D. P. Bertsekas, J. N. Tsitsiklis, “Introduction to Probability”, 2nd Ed., Athena Science 2008.

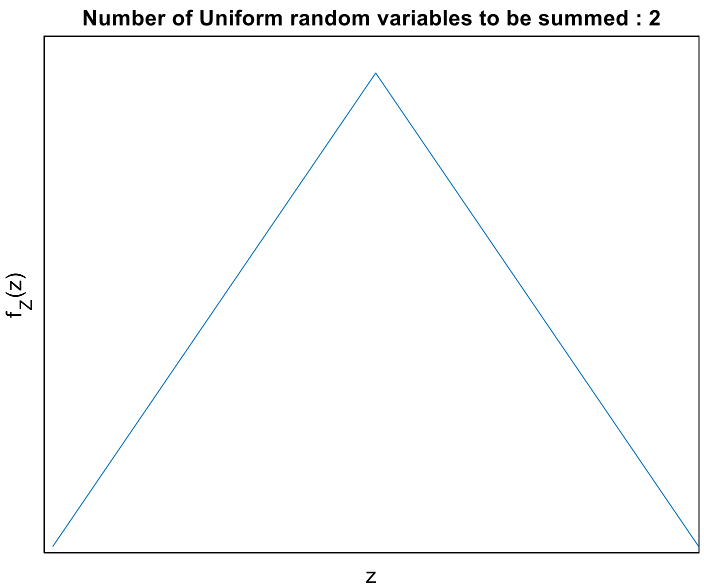
Textbooks: Fikri Öztürk, Levent Özbek, “Matematiksel Modelleme ve Simülasyon”, 2004.
Averill M. Law, “Simulation Modeling and Analysis”, McGraw-Hill, 2015.

Sum of Independent Random Variables – Convolution

Similar result can be obtained for the continuous case such as (see page 213 in the textbook, *Bertsekas*, 2008)

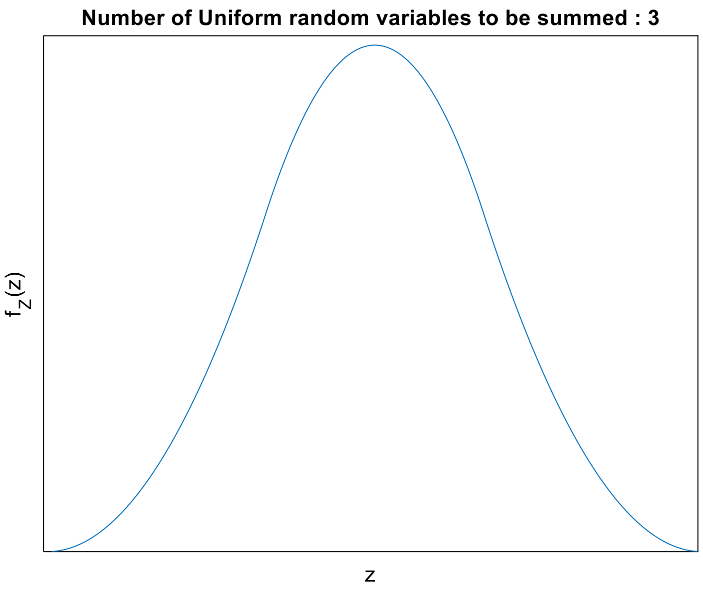
$$f_Z(z) = \int_{-\infty}^{\infty} f_{XZ}(x,z)\,dx = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\,dx = f_X(x) * f_Y(y)$$

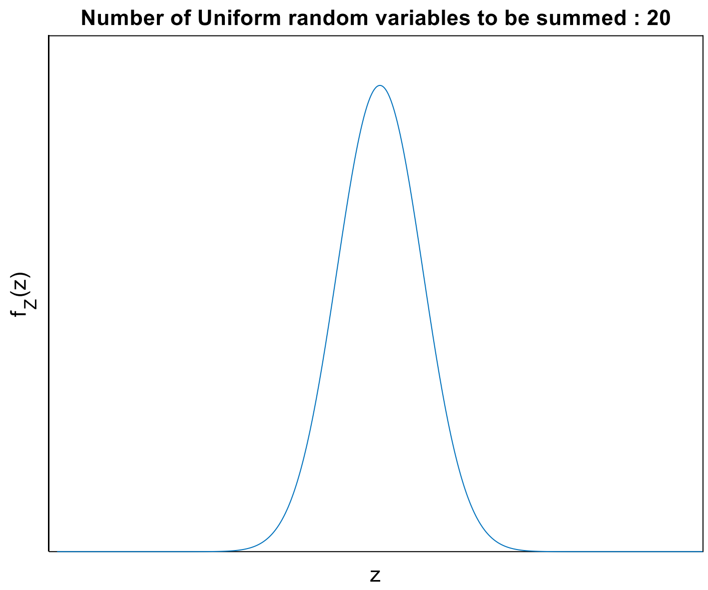




Textbooks: Fikri Öztürk, Levent Özbek, “Matematiksel Modelleme ve Simülasyon”, 2004.
Averill M. Law, “Simulation Modeling and Analysis”, McGraw-Hill, 2015.

Sum of Independent Random Variables – Convolution





$$p_Z(z) = p_X(x) * p_Y(y) = \sum_x p_X(x)\, p_Y(z-x)$$

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

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Averill M. Law, “Simulation Modeling and Analysis”, McGraw-Hill, 2015.

Central Limit Theorem

Let $X_1,\ldots,X_n$ be a sequence of **i.i.d.** random variables with mean μ and variance σ^2

$$Z_n = \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

CDF of Z_n converges to the standart Gaussian CDF

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(x^2/2\right) dx,$$

in the sense that

$$\lim_{n\to\infty} P\left(Z_n \le z\right) = \Phi(z), \quad \text{for every } z.$$

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Textbooks: Fikri Öztürk, Levent Özbek, “Matematiksel Modelleme ve Simülasyon”, 2004.
Averill M. Law, “Simulation Modeling and Analysis”, McGraw-Hill, 2015.

Simulation: Sum of Uniform Random Variables

Generating random numbers from a uniformly distributed random variables over the interval (0,1), i.e. $X_i \sim U(0,1)$

Number of random variables to be summed: 20

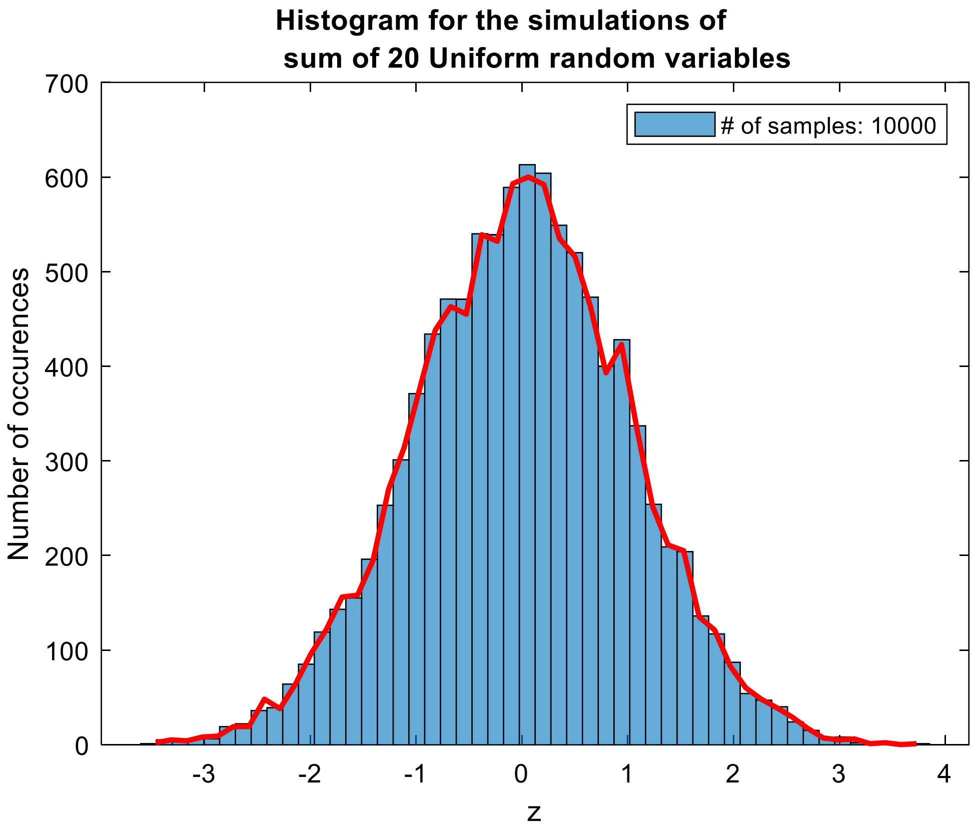
Number of generated random samples: 10 and 10000

Note that

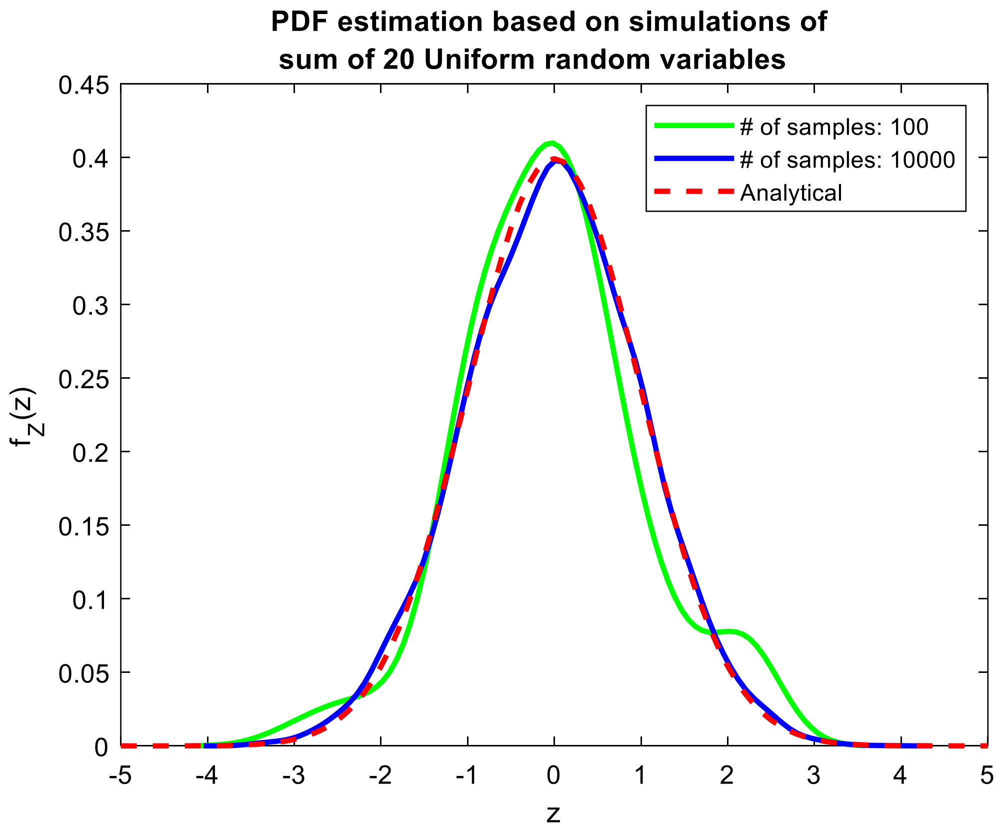
- the interval can be either closed ([a, b]) or open ((a, b)).
- Standard deviation and mean of a uniform random variable over the interval (0,1) are $\sqrt{1/12}$ and $1/2$, respectively.

Textbooks: Fikri Öztürk, Levent Özbek, “Matematiksel Modelleme ve Simülasyon”, 2004.
Averill M. Law, “Simulation Modeling and Analysis”, McGraw-Hill, 2015.

Simulation: Sum of Uniform Random Variables



Simulation: Sum of Uniform Random Variables



Relation Between Two Random Variables

Textbooks: Fikri Öztürk, Levent Özbek, “Matematiksel Modelleme ve Simülasyon”, 2004.
Averill M. Law, “Simulation Modeling and Analysis”, McGraw-Hill, 2015.

The "**correlation coefficient"** of two random variables X and Y:

$$\rho_{XY} = E\left[\left(\frac{X - E[X]}{\sigma_X}\right)\left(\frac{Y - E[Y]}{\sigma_Y}\right)\right] = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

Measure of linear correlation (statistical relationship between two variables)

$$-1 \le \rho \le 1$$

- $\rho_{XY} = 0$: Uncorrelated (no linear relationship)
  (does not imply **independence**)
- $\rho_{XY} = 1$: perfect positive linear relationship

$$\frac{Y - E[Y]}{\sigma_Y} = \frac{X - E[X]}{\sigma_X}$$

- $\rho_{XY} = -1$: perfect positive linear relationship

$$\frac{Y - E[Y]}{\sigma_Y} = (-)\frac{X - E[X]}{\sigma_X}$$