DIRECTIONAL DERIVATIVES

To this point we've only looked at the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$. Recall that these derivatives represent the rate of change of f as we vary x(holding y fixed) and as we vary y (holding x fixed) respectively. We now need to discuss how to find the rate of change of f if we allow both x and y to change simultaneously. The problem here is that there are many ways to allow both x and y to change. For instance, one could be changing faster than the other and then there is also the issue of whether or not each is increasing or decreasing. So, before we get into finding the rate of change we need to get a couple of preliminary ideas taken care of first. The main idea that we need to look at is just how are we going to define the changing of x and/or y.

Let f be defined in a neighborhood of a point $P_0 = (x_0, y_0)$ $[P_0 = (x_0, y_0, z_0)]$ and let l be the directed line through P_0 with direction vector $\vec{u} = (u_1, u_2)$ or $\left[\vec{u} = (u_1, u_2, u_3)\right]$ $\left(l\vec{u}\right)$. Then l has the parametric equations in $\mathbb{R}\#\mathbb{I}^{\not\vDash}$

$$cl : x = x_0 + tu_1$$

y=y$_0 + tu_2 \# I \# I$

or in $\mathbb{R}^3$

$$\begin{aligned} l:\ \ x &= x_0 + tu_1 \\ y &= y_0 + tu_2 \\ z &= z_0 + tu_3 \end{aligned}$$

where t is a parameter. By a directional derivative of a function $f(x,y)$ (or $f(x,y,z)$) at $P_0 = (x_0, y_0)$ [or $P_0 = (x_0, y_0, z_0)$] in the direction of the unit vector $\vec{u} = (u_1, u_2)$ $\left[\text{or } \vec{u} = (u_1, u_2, u_3)\right]$ on the line l, denoted by

$$\begin{aligned} D_{\vec{u}} f(P_0) &= D_{\vec{u}} f(x_0, y_0) \text{ or} \\ D_{\vec{u}} f(P_0) &= D_{\vec{u}} f(x_0, y_0, z_0). \end{aligned}$$

We mean the limit

$$D_{\vec{u}} f(P_0) = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0, z_0)}{t}$$

or

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3) - f(x_0, y_0)}{t}$$

proved that this exists and is finite.

Definition 127 (Gradient Vector) *The gradient (or gradient vector) of $f(x,y,z)$ at a point $P_0 = (x_0, y_0, z_0)$ is the vector*

$$\begin{aligned} (\nabla f)_{P_0} &= \nabla f(P_0) \\ &= f_x(P_0)\, i + f_y(P_0)\, j + f_z(P_0)\, k \\ &= \frac{\partial f}{\partial f}(P_0)\, i + \frac{\partial f}{\partial y}(P_0)\, j + \frac{\partial f}{\partial z}(P_0)\, k \end{aligned}$$

Theorem 128 (Directional Derivative in terms of Partial Derivatives) *If $f(x,y,z)$ is differentiable at $P_0 = (x_0, y_0, z_0)$ and $\overrightarrow{u} = (u_1, u_2, u_3)$ is aunit vector, then the directional derivative of f at P_0 in the direction of u is given by*

$$\begin{aligned}
D_{\overrightarrow{u}} f(P_0) &= f_x(P_0)\,u_1 + f_y(P_0)\,u_2 + f_z(P_0)\,u_3 \\
&= (f_x(P_0), f_y(P_0), f_z(P_0))\,(u_1, u_2, u_3) \\
&= (\nabla f)_{P_0}\,\overrightarrow{u}
\end{aligned}$$

Example 129 *Find the directional derivative of $f(x,y) = x^2y + xy^2 + 3$ at the point $P(1,2)$ in the direction of the unit vector $\overrightarrow{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.*

Solution 130

$$\begin{aligned}
f_x &= 2xy + y^2 \\
f_y &= x^2 + 2xy \\
(\nabla f) &= \left(2xy + y^2\right) i + \left(x^2 + 2xy\right) j
\end{aligned}$$

$$\begin{aligned}
D_{\overrightarrow{u}} f(x,y,z) &= (\nabla f)\,\overrightarrow{u} \\
&= \left(2xy + y^2\right)\frac{1}{\sqrt{2}} + \left(x^2 + 2xy\right)\frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
D_{\overrightarrow{u}} f(1,2) &= (\nabla f)_{P_0}\,\overrightarrow{u} \\
&= \left(2.1.2 + 2^2\right)\frac{1}{\sqrt{2}} + \left(1^2 + 2.1.2\right)\frac{1}{\sqrt{2}} \\
&= \frac{13}{\sqrt{2}}
\end{aligned}$$

Example 131 *Find the directional derivative of the function given by $f(x,y,z) = x^2z + xyz + y^3 + 3$ in the direction of the line associated the point $P(2,-2,1)$ to the point $Q(3,2,-2)$. Then find the value of this derivative at the point P.*

Solution 132 $\overrightarrow{PQ} = (3,2,-2) - (2,-2,1) = (1,4,-3)$ *This is not a unit vector.*

$$u = \frac{PQ}{\|PQ\|} = \frac{(1,4,-3)}{\sqrt{1+16+9}} = \left(\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, -\frac{3}{\sqrt{26}}\right)$$

Now u is aunit vector which is parallet tu PQ.

$$\begin{aligned}
D_{\overrightarrow{u}} f &= \nabla f\,\overrightarrow{u} = \frac{1}{\sqrt{26}} f_x + \frac{4}{\sqrt{26}} f_y - \frac{3}{\sqrt{26}} f_z \\
&= \frac{1}{\sqrt{26}}(2xz + yz) + \frac{4}{\sqrt{26}}\left(xz + 3y^2\right) - \frac{3}{\sqrt{26}}\left(x^2 + xy\right)
\end{aligned}$$

$$\begin{aligned} D_{\vec{u}} f(2,-2,1) &= \frac{1}{\sqrt{26}}(4-2) + \frac{4}{\sqrt{26}}(2+12) - \frac{3}{\sqrt{26}}(4-4) \\ &= \frac{58}{\sqrt{26}} \end{aligned}$$

Theorem 133 *The maximum value of $D_{\vec{u}} f$ (and hence then the maximum rate of change of the function $f(x)$) is given by $\| \nabla f(x) \|$ and will occur in the direction given by $\nabla f(x)$.*

Proof. This is a really simple proof. First, if we start with the dot product form $D_{\vec{u}} f$ and use a nice fact about dot products as well as the fact that u is a unit vector we get,

$$D_{\vec{u}} f = \nabla f \cdot u = \| \nabla f \| \| u \| \, cos\theta = \| \nabla f \| \, cos\theta$$

where θ is the angle between the gradient and u . Now the largest possible value of $cos\theta$ is 1 which occurs at $\theta = 0$. Therefore the maximum value of $D_{\vec{u}} f(x)$ is $\| \nabla f(x) \|$ Also, the maximum value occurs when the angle between the gradient and u is zero, or in other words when u is pointing in the same direction as the gradient, $\nabla f(x)$. ■

Remark 134 *Let assume an unit vector u in a three dimensional space and suppose that α, β and γ are the angels between coordinate axises and u respectively. Then the components of u are given $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ respectively.These components are called directrix cosinuses. So the directional derivative of $w = f(x,y,z)$ in the direction of the vector u can be written by*

$$D_u f = f_x \cos\alpha + f_y \cos\beta + f_z \cos\gamma.$$

If the function $w = f(x,y)$ is a two variables function then, the directional derivative in the direction of the vector u can be written by

$$D_u f = f_x \cos\alpha + f_y \cos\beta$$

where α is the angle between u and $x-axis$ and similarly β is the angle between u and $y-axis$. So $\beta = \left(\frac{\pi}{2} - \alpha\right)$ and we get

$$\begin{aligned} D_u f &= f_x \cos\alpha + f_y \cos\left(\frac{\pi}{2} - \alpha\right) \\ &= f_x \cos\alpha + f_y \sin\alpha \end{aligned}$$

Example 135 *Find the directional derivative of $f(x,y) = x^2 y + xy^2 + 3$ in the direction of $\alpha = \pi/3$.*

Solution 136

$$\begin{aligned} D_u f &= f_x \cos\alpha + f_y \sin\alpha \\ &= \left(2xy + y^2\right)\cos\frac{\pi}{3} + \left(x^2 + 2xy\right)\sin\frac{\pi}{3} \\ &= \frac{1}{2}\left(2xy + y^2\right) + \frac{\sqrt{3}}{2}\left(x^2 + 2xy\right) \end{aligned}$$

Example 137 *Find the largest and smallest value of the directional derivative of the function* $f(x, y, z) = x^2 + y^2 - z^2$ *at the point* $P(1, 2, 3)$.

Solution 138 $\nabla f(x, y, z) = (2x, 2y, -2z)$ *and* $\nabla f(1, 2, 3) = (2, 4, -6)$. *The maximum value of directional derivative is*

$$\|\nabla f(1, 2, 3)\| = \sqrt{2^2 + 4^2 + (-6)^2} = \sqrt{56} = 2\sqrt{14}$$

in the direction of $\nabla f(1, 2, 3) = (2, 4, -6)$. *And minimum value is* $-2\sqrt{14}$ *in the direction of* $-\nabla f(1, 2, 3) = (2, 4, -6)$