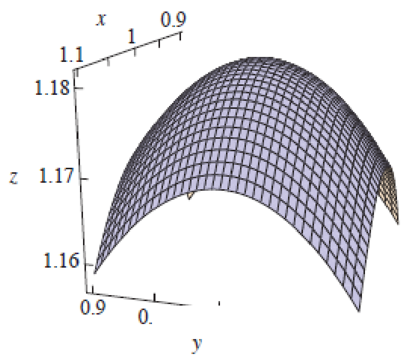
8.5 EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

In this section we are going to extend one of the more important ideas from Analysis I into functions of two variables. We are going to start looking at trying to find minimums and maximums of functions. This in fact will be the topic of the following two sections as well.

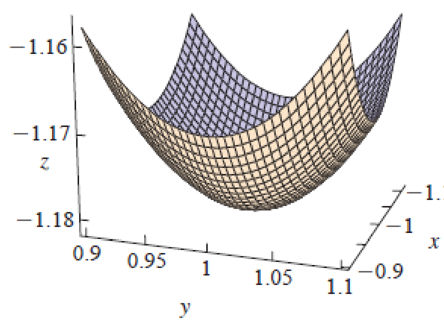
Definition 142 a *A function $f(x, y)$ has a local (relative) minimum at the point (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in some region around (a, b) .*

b *A function $f(x, y)$ has a local (relative) maximum at the point (c, d) if $f(x, y) \leq f(c, d)$ for all points (x, y) in some region around (c, d) .*

Local maxima and minima are also called local extrama.



Local maximum



Local minimum

Note that this definition does not say that a local minimum is the smallest value that the function will ever take. It only says that in some region around the point (a, b) the function will always be larger than $f(a, b)$. Outside of that region it is completely possible for the function to be smaller. Likewise, a local maximum only says that around (c, d) the function will always be smaller than $f(a, b)$. Again, outside of the region it is completely possible that the function will be larger. Next, we need to extend the idea of critical points up to functions of two variables. Recall that a critical point of the function $f(x)$ was a number $x = c$ so that either $f'(c) = 0$ or $f'(c)$ doesn't exist. We have a similar definition for critical points of functions of two variables.

Theorem 143 *If $f(x, y)$ has a local extrama at an interior point (a, b) of its domain and if the first partial derivatives exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$*

From this theorem we conclude that the only places a function $f(x, y)$ can have an extrama value if;

1. Interior points where $f_x = f_y = 0$
2. Interior points where at least one of the partial derivatives f_x or f_y does not exist
3. Boundary points of the function f.

Definition 144 *The point (a,b) is a critical point (or a stationary point) of $f(x,y)$ provided one of the following is true,*

a $\nabla f(a,b) = 0$ *(this is equivalent to saying that $f_x(a,b) = 0$ and $f_y(a,b) = 0$),*

b $f_x(a,b)$ *and/or* $f_y(a,b)$ *doesn't exist.*

Theorem 145 *Suppose that (a,b) is a critical point of $f(x,y)$ and that the second order partial derivatives are continuous in some region that contains (a,b) . Next define,*

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

We then have the following classifications of the critical point.

1. *$D > 0$ and $f_{xx}(a,b) > 0$ then there is a local minimum at (a,b) .*
2. *$D > 0$ and $f_{xx}(a,b) < 0$ then there is a local maximum at (a,b) .*
3. *If $D < 0$ then the point (a,b) is a saddle point.*
4. *If $D = 0$ then the point (a,b) may be a local minimum, local maximum or a saddle point. Other techniques would need to be used to classify the critical point.*

Remark 146 *Note that if $D > 0$ then both $f_{xx}(a,b)$ and $f_{yy}(a,b)$ will have the same sign and so in the first two cases above we could just as easily replace $f_{xx}(a,b)$ with $f_{yy}(a,b)$.*

Example 147 *Find and classify all the critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$.*

Solution 148 *We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so let's get those.*

$$\begin{aligned} f_x &= 3x^2 - 3y \text{ and } f_y = 3y^2 - 3x \\ f_{xx} &= 6x \text{ , } f_{yy} = 6y \\ f_{xy} &= -3 \end{aligned}$$

Let's first find the critical points. Critical points will be solutions to the system of equations,

$$\begin{aligned} f_x &= 3x^2 - 3y = 0 \\ f_y &= 3y^2 - 3x = 0 \end{aligned}$$

We can solve the first equation for y as follows

$$3x^2 - 3y = 0 \Rightarrow y = x^2$$

Plugging this into the second equation gives,

$$3(x^2)^2 - 3x = 3x(x^3 - 1) = 0$$

From this we can see that we must have $x = 0$ or $x = 1$. Now use the fact that $y = x^2$ to get the critical points.

$$\begin{array}{lll} x = 0, \quad y = 0 & \Rightarrow & (0,0) \\ x = 1, \quad y = 1 & \Rightarrow & (1,1) \end{array}$$

So, we get two critical points. All we need to do now is classify them

$$\begin{aligned} D(x,y) &= f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 \\ &= (6x)(6y) - (-3)^2 \\ & \quad 36xy - 9 \end{aligned}$$

$(0,0)$

$$D(0,0) = -9 < 0$$

So, for $(0,0)$ D is negative and so this must be a saddle point.
$(1,1)$

$$D(1,1) = 36 - 9 = 27 > 0 \text{ and } f_{xx}(1,1) = 6 > 0$$

For $(1,1)$ D is positive and f_{xx} is positive and so we must have a local minimum.

8.5.1 Absolute Extrema

In this section we want to optimize a function, that is identify the absolute minimum and/or the absolute maximum of the function, on a given region in $\mathbb{R}^2$. Absulute maximum value is the biggest value that the function f attains on a region. Absolute minimum value is the minimal (smalest) value that the function f attains on o region.

Definition 149 *We call $f(a,b)$ the absolute maximum of f on the region R if $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$. Similarly, $f(a,b)$ is called the absolute minimum of f on R if $f(a,b) \leq f(x,y)$ for all $(x,y) \in R$. In either case, $f(a,b)$ is called an absolute extremum of f.*

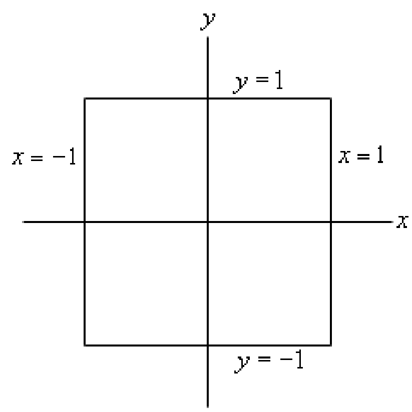
Theorem 150 *If $f(x,y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D , (x_1,y_1) and (x_2,y_2) so that $f(x_1,y_1)$ is the absolute maximum and $f(x_2,y_2)$ is the absolute minimum of the function in D .*

Finding Absolute Extrema

1. Find all the critical points of the function that lie in the region D and determine the function value at each of these points.

2. Find all extrema of the function on the boundary.
3. The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

Example 151 *Find the absolute minimum and absolute maximum of* $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ *on the rectangle given by* $-1 \leq x \leq 1$ *and* $-1 \leq y \leq 1$



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Solution 152 *The boundary of this rectangle is given by the following conditions.*

Solution 153 *right side :*

$$\begin{aligned} right\ \ side &: x = 1,\ -1 \leq y \leq 1 \\ left\ side &: x = -1,\ -1 \leq y \leq 1 \\ upper\ side &: y = 1,\ -1 \leq x \leq 1 \\ lower\ side &: y = -1,\ -1 \leq x \leq 1 \end{aligned}$$

$$\begin{aligned} f_x &= 2x - 4xy \\ f_y &= 8y - 2x^2 \end{aligned}$$

To find the critical points we will need to solve the system

$$\begin{aligned} 2x - 4xy &= 0 \\ 8y - 2x^2 &= 0 \\ y &= \frac{x^2}{4} \end{aligned}$$

Plugging this into the first equation gives us,

$$\begin{aligned} x(2 - x^2) &= 0 \\ x &= 0,\ x = \pm\sqrt{2} \end{aligned}$$

Now, recall that we only want critical points in the region that we're given. That means that we only want critical points for which $-1 \leq x \leq 1$ *. The only value of* x *that will satisfy this is the first one so we can ignore the last two for this problem. Note however that a simple change to the boundary would include these two so don't forget to always check if the critical points are in the region (or on the boundary since that can also happen). The single critical point, in the region (and again, that's important), is (0,0). We now need to get the value of the function at the critical point.*

$$f(0,0) = 4$$

Eventually we will compare this to values of the function found in the next step and take the largest and smallest as the absolute extrema of the function in the rectangle. Now we have reached the long part of this problem. We need to find the absolute extrema of the function along the boundary of the rectangle. What this means is that we're going to need to look at what the function is doing along each of the sides of the rectangle listed above. Let's first take a look at the right side. As noted above the right side is defined by

$$x = 1,\ -1 \leq y \leq 1$$

$$g(y) = f(1,y) = 1^2 + 4y^2 - 2\left(1^2\right)y + 4 = 5 + 4y^2 - 2y$$

Now, finding the absolute extrema of $f(x,y)$ *along the right side will be equivalent to finding the absolute extrema of* $g(y)$*in the range* $-1 \leq y \leq 1$.

$$g^{'}(y) = 8y - 2 \Rightarrow y = \frac{1}{4}$$

Notice that, using the definition of $g(y)$ *these are also function values for* $f(x,y)$

$$\begin{aligned} g(-1) &= f(1,-1) = 11 \\ g(1) &= f(1,1) = 7 \\ g\left(\frac{1}{4}\right) &= f\left(1,\frac{1}{4}\right) = \frac{19}{4} = 4.75 \end{aligned}$$

We can now do the left side, upper side and lower side of the rectangle similarly. The final step to this process is to collect up all the function values for $f(x,y)$ *that we've computed in this problem. Here they are,*

$$\begin{array}{ccc} f(0,0) = 4 & f(1,-1) = 11 & f(1,1) = 7 \\ f\left(1,\frac{1}{4}\right) = 4.75 & f(-1,1) = 7 & f(-1,-1) = 11 \\ f\left(-1,\frac{1}{4}\right) = 4.75 & f(0,1) = 8 & f(0,-1) = 8 \end{array}$$

The absolute minimum is at $(0,0)$ *since gives the smallest function value and the absolute maximum occurs at* $(1,-1)$ *and* $(-1,-1)$ *since these two points give the largest function value.*