8.8.2 Region Transformations For Double Integrals

As we know, the region D given in $uv-$coordinate system can be written in $xy-$coordinate system by using $x = g(u,v)$ and $y = h(u,v)$ as

$$\iint_R f(x,y)\,dxdy = \iint_R f(g(u,v), h(u,v))\,|J|\,dudv$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0.$$

If we use special transformation $x = r\cos\theta$ and $y = r\sin\theta$ for changing the regions by using polar coordinates we have

$$\iint_R f(x,y)\,dxdy = \iint_R f(r\cos\theta, r\sin\theta)\,rdrd\theta.$$

The formula in above says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $rdrd\theta$.

Theorem 185 *If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$ where $0 \le \beta - \alpha \le 2\pi$ then*

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

Theorem 186 *If f is continuous on a polar region of the form*

$$R = \{(r,\theta)\} : \alpha \le \theta \le \beta,\ \ h_1(\theta) \le r \le h_2(\theta)$$

then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,rdrd\theta.$$

Example 187 *Evaluate $\iint_R (3x+y)\,dA$ where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=9$.*

Solution 188 *The region R can be described as*

$$R = \left\{(x,y) : y \ge 0,\ \ 1 \le x^2+y^2 \le 9\right\}$$

$$\begin{aligned}
\iint\limits_R (3x+y)\,dA &= \int_0^{\pi}\int_1^{3} (3r\cos\theta + r\sin\theta)\,r dr d\theta = \int_0^{\pi}\int_1^{3}\left(3r^2\cos\theta + r^2\sin\theta\right) dr d\theta \\
&= \int_0^{\pi}\left(r^3\cos\theta + \frac{r^3}{3}\sin\theta\right)\Bigg|_{r=1}^{r=3} d\theta = \int_0^{\pi}\left(26\cos\theta + \frac{26}{3}\sin\theta\right) d\theta \\
&= \left(26\sin\theta - \frac{26}{3}\cos\theta\right)\Bigg|_{\theta=0}^{\theta=\pi} = \frac{52}{3}
\end{aligned}$$

Example 189 *Let R is the region bounded by the lines $y = 2x - 2,\ y = 2x - 4,\ y = -x + 2$ and $y = -x + 4$. The region can be converted the new region D by using the transformation $u = x + y$, $v = 2x - y$. Evaluate the integral*

$$\iint\limits_R (2x-y)^2\, dxdy.$$

Solution 190 $\begin{array}{rcl} x+y=2 & \Longrightarrow & u=2 \\ x+y=4 & \Longrightarrow & u=4 \\ 2x-y=2 & \Longrightarrow & v=2 \\ 2x-y=4 & \Longrightarrow & v=4 \end{array}$ *then* $J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{-1}{3}$.

$$\iint\limits_R (2x-y)^2\, dxdy = \iint\limits_D v^2\,|-1/3|\, dudv = \frac{1}{3}\int_2^4\int_2^4 v^2 dudv = \frac{112}{9}$$

8.9 APPLICATIONS OF DOUBLE INTEGRALS

In this section we will give some applications of double integrals which are necessary for Physics, Mathematics and Engineering.

8.9.1 AREAS

Double integrals can be used to evaluate areas. It is known from the defenition of double integrals that if f is a continuous function defined on a bounded region R, i.e.,$f : R \subset \mathbb{R}^2 \to \mathbb{R}$, $P = \{R_1, R_{2,} \cdots, R_n\}$ is a partition of R and ΔA_i is the area of R_i $(i = 1, 2, \cdots, n)$, then for (x_i, y_i)

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i = \iint_R f(x, y)\,dA.$$

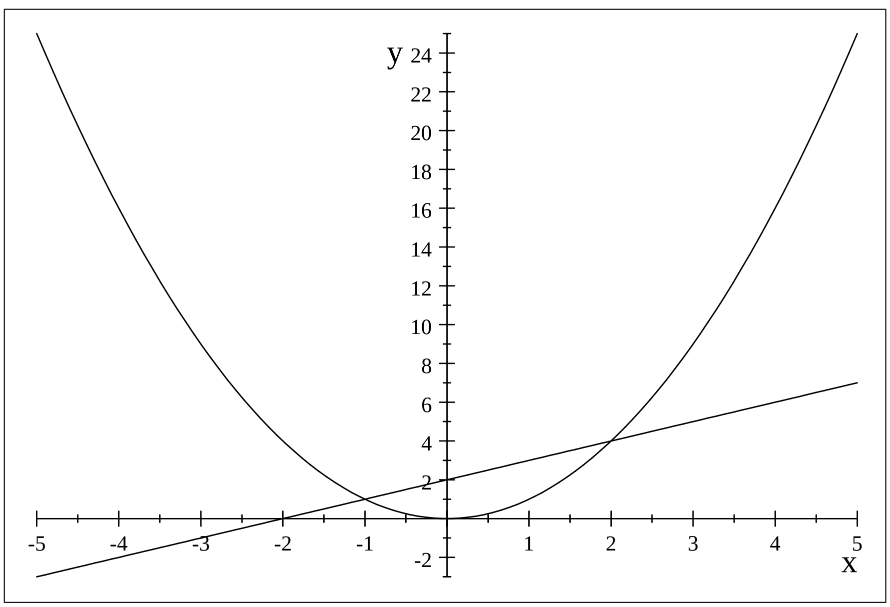
If we take $f(x, y) = 1$ for all $(x, y) \in R$, we get

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \Delta A_i = \iint_R dA$$

where

$$\text{Area}(R) \cong \sum_{i=1}^{n} \Delta A_i.$$

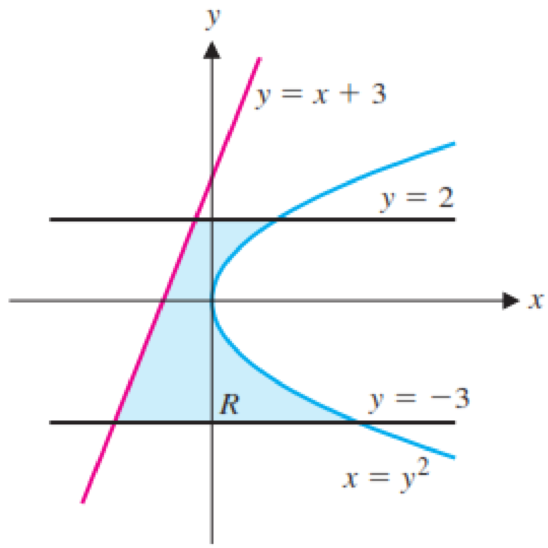
Example 191 *Find the area of R bounded by the curve $y = x^2$ and the line $y = x + 2$ by using double integral.*



Solution 192

$$A = \iint_R dxdy = \int_{-1}^{2}\int_{x^2}^{x+2} dydx = \int_{-1}^{2} \left(x + 2 - x^2\right) dx = \frac{7}{6}\ units^2$$

Example 193 *Find the area of the plane region bounded by the graphs of* $x = y^2$*,*$y - x = 3$*,*$y = -3$ *and* $y = 2$*.*

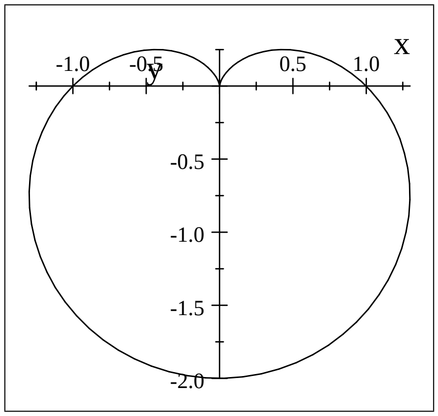


Solution 194

$$\begin{aligned} A &= \iint_R dxdy = \int_{-3}^{2}\int_{y-3}^{y^2} dxdy = \int_{-3}^{2} \left[y^2 - (y-3)\right] dy \\ &= \left[\frac{y^3}{3} - \frac{y^2}{2} + 3y\right]_{-3}^{2} = \frac{175}{6} \ units^2 \end{aligned}$$

Example 195 *Find the area of region bounded by the cardioid* $r = 1 - \sin\theta$*.*

Solution 196



$$\begin{aligned}
A &= \iint_R r dr d\theta = 2\int_{-\pi/2}^{\pi/2}\int_0^{1-\sin\theta} r dr d\theta = 2\int_{-\pi/2}^{\pi/2}\frac{(1-\sin\theta)^2}{2}d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(1-2\sin\theta+\sin^2\theta\right)d\theta = \int_{-\pi/2}^{\pi/2}\left(1-2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta \\
&= \frac{3\pi}{2}\ units^2
\end{aligned}$$

8.9.2 VOLUMES

Let f be a continuous and postive function on R (For every $(x,y)\in R$, $f(x,y)\geq 0$). Then the expression

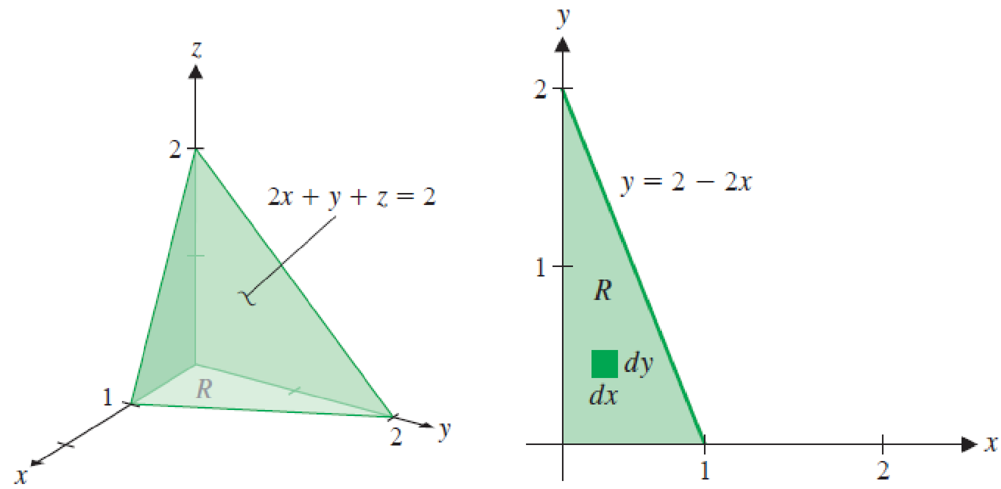
$$\sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$$

is the sums of the volumes of cylenders whose base area is ΔA_k and height is $f(x_k,y_k)$.

$$V = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k = \iint_R f(x,y)\,dxdy$$

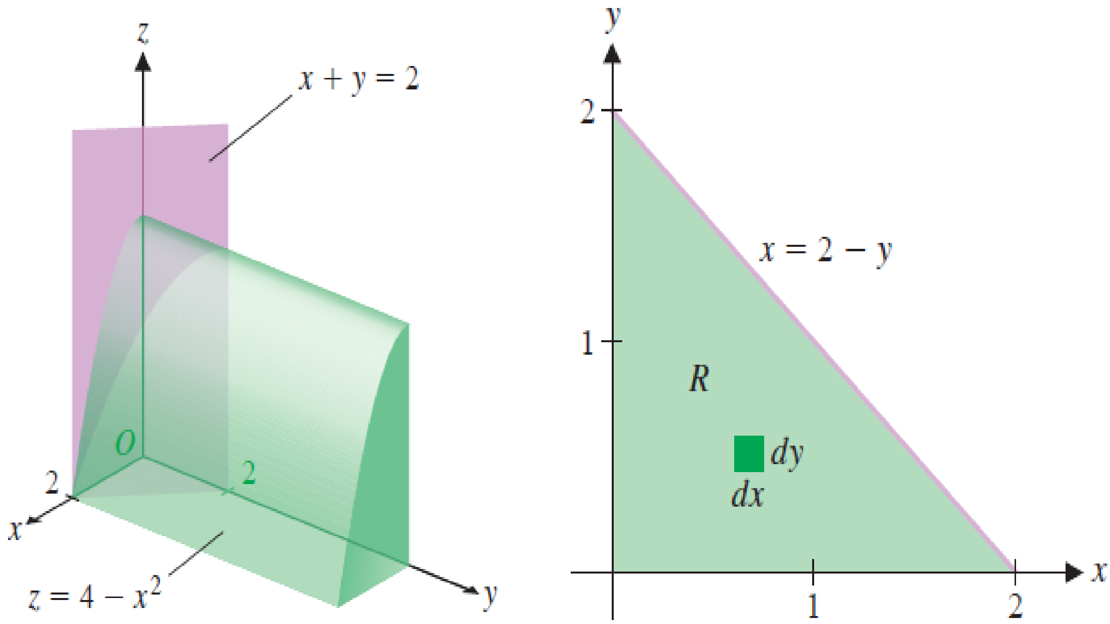
Example 197 *Find the volume of the tetrahedron bounded by the plane* $2x+y+z=2$ *and the three coordinate planes.*

Solution 198 *The plane* $2x+y+z=2$ *intersects the coordinate axes at the points* $(1,0,0)$, $(0,2,0)$ *and* $(0,0,2)$.



$$V = \int_0^1 \int_0^{2-2x} (2 - 2x - y)\, dy dx = \frac{2}{3}\ units^3$$

Example 199 *Find the volume of the solid lying in the first octant and bounded by the graphs of* $z = 4 - x^2$*,*$x + y = 2$*,* $x = 0$*,* $y = 0$ *and* $z = 0$*.*



$$V = \int_0^2 \int_0^{2-y} \left(4 - x^2\right) dxdy = \frac{20}{3} \ units^3$$

Example 200 *Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.*

Solution 201 *For $z = 0$ in the equation of the paraboloid, we get $1 = x^2 + y^2$.*