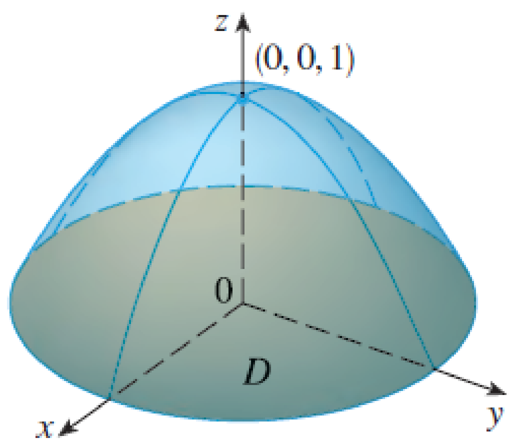


$$
\begin{aligned}
V &= \iint_R \left(1 - x^2 - y^2\right) dxdy = \int_0^{2\pi}\int_0^1 \left(1 - r^2\right) r dr d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(r - r^3\right) dr d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \frac{\pi}{2}\ units^3
\end{aligned}
$$

8.9.3 MASS

We were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density. But now, equipped with the double integral, we can consider a lamina with variable density. Suppose the lamina occupies a region R and its density (in units of mass per unit area) at a point (x, y) in R given by $\rho(x, y)$ where ρ s a continuous function on R.

Let P is a partition of R

$$P = \{B_1, B_2, \cdots, B_n\}$$

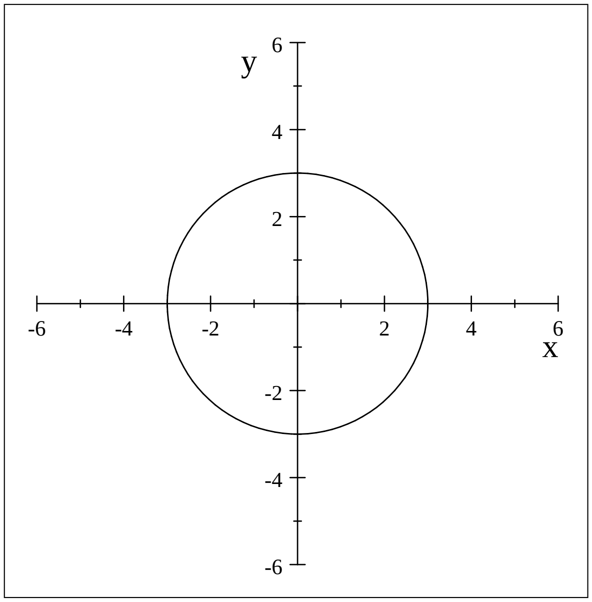
and ΔA_k is the area of B_k for $k = 1, , 2, \cdots, n$. If we choose a point $(x_k, y_k) \in B_k$, then the mass of the part of the lamina that occupies B_k is approximately $\rho(x_k, y_k)\Delta A_k$. If we add all such masses, we get an approximation to the total mass:

$$M \cong \sum_{k=1}^{n} \rho(x_k, y_k)\Delta A_k$$

If we now increase the number of subrectangles, we obtain the total mass M of the lamina as the limiting value of the approximations:

$$M = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \rho(x_k, y_k)\Delta A_k = \iint_R \rho(x, y)\, dA$$

Example 202 *Let R is an unit circle with a radius $r = 3$. Suppose that the density of this plate at eachpoint changes proportionally with the distance from the center point ofthe circle. If the density of boundary of circle is 6 then evaluate the mass of this plate.*



$\rho(x, y) = \alpha\sqrt{x^2 + y^2},$

$|OP| = \sqrt{x^2 + y^2}$

For $\sqrt{x^2 + y^2} = 9$ $\rho(x, y) = 6$ then $6 = \alpha 3 \Rightarrow \alpha = 2$ and so $\rho(x, y) = 2\sqrt{x^2 + y^2}$.

$$M = \iint_R \rho(x, y)\, dxdy = \int\limits_{x^2+y^2 \le 9}\int 2\sqrt{x^2 + y^2} dxdy$$

For $x = r\cos\theta$, $y = r\sin\theta$ and $|J| = r$, $R = \{(r, \theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

$$M = \int_0^{2\pi}\int_0^3 2\sqrt{r^2} r dr\theta = \int_0^{2\pi} \frac{2r^3}{3}\Big|_{r=0}^{r=3} d\theta = 18\theta\Big|_{\theta=0}^{\theta=2\pi} = 36\pi\ units^2$$

8.9.4 CENTER OF MASS

The center of mass or centroid of a region is the point in which the region will be perfectly balanced horizontally if suspended from that point. Let suppose a plate in a region R and this plate has density $\rho(x, y)$ at the point (x, y). Let P is apartition of R and ΔR_k is an area of R_k $(k = 1, , 2, \cdots, n)$. If the partition for P is thin enough then it can be considered that the mass accumulateson the point (x_k, y_k). These points are called mass point and the coordinates of the center of mass can be given,

$$\bar{x}=\frac{\iint\limits_R x\rho(x,y)\,dxdy}{\iint\limits_R \rho(x,y)\,dxdy},\ \bar{y}=\frac{\iint\limits_R y\rho(x,y)\,dxdy}{\iint\limits_R \rho(x,y)\,dxdy}$$

or

$$\begin{aligned}\bar{x} &= \frac{1}{M}\iint\limits_R x\rho(x,y)\,dxdy\\ \bar{y} &= \frac{1}{M}\iint\limits_R y\rho(x,y)\,dxdy\end{aligned}$$

If the density of this plate is constant, since $M=\rho A$, then

$$\begin{aligned}\bar{x} &= \frac{1}{A}\iint\limits_R xdxdy\\ \bar{y} &= \frac{1}{A}\iint\limits_R ydxdy\end{aligned}$$

Example 203 *Let suppose that R is a region bounded by the curve $\sqrt{x}+\sqrt{y}=1$ and the coordinate axis and suppose a plate in R. Then find the coordinates of the center of mass of this plate (The plate is homogeneous, the density is constant on each point).*

Solution 204 *First we will find the area of R.*

$$\begin{aligned}A &= \iint\limits_R dxdy=\int_0^1\int_0^{\left(1-\sqrt{x}\right)^2} dydx=\frac{1}{6}\\ \bar{x} &= \frac{1}{\frac{1}{6}}\iint\limits_R xdxdy=6\int_0^1\int_0^{\left(1-\sqrt{x}\right)^2} xdydx=\frac{1}{5}\\ \bar{y} &= \frac{1}{\frac{1}{6}}\iint\limits_R ydxdy=6\int_0^1\int_0^{\left(1-\sqrt{x}\right)^2} ydydx=\frac{1}{5}\end{aligned}$$

The coordinates of the center of mass is $\left(\frac{1}{5},\frac{1}{5}\right)$

Example 205 *Find the center of mass of a thin, uniform plate whose shape is the region between $y = cosx$ and the $x-$axis between $x = -\pi/2$ and $x = \pi/2$. Since the density is constant, we may take $\rho(x, y) = 1$.*

Solution 206 *It is clear that $\bar{x} = 0$, but for practice let's compute it anyway. First we compute the mass:*

$$M == A = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} 1\, dy\, dx = \int_{-\pi/2}^{\pi/2} \cos x\, dx = \sin x \mid_{-\pi/2}^{\pi/2} = 2$$

$$M_y = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y\, dy\, dx = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos^2 x\, dx = \frac{\pi}{4}$$

$$M_x = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} x\, dy\, dx = \int_{-\pi/2}^{\pi/2} x \cos x\, dx = 0$$

$$\bar{x} = 0 \text{ and } \bar{y} = \frac{M_y}{M} = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}$$

Exercise 207 *Find the center of mass of a two-dimensional plate that occupies the square $[0, 1] \times [0, 1]$ and has density function xy.*

Exercise 208 *Find the center of mass of a two-dimensional plate that occupies the triangle $0 \le x \le 1, 0 \le y \le x$, and has density function xy.*

Exercise 209 *Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 0, y = x$, and $2x + y = 6$ and has density function x^2.*