8.10 TRIPLE INTEGRALS

We can also do triple integrals—integrals over a three-dimensional region. The simplest application allows us to compute volumes in an alternate way.

We first consider the relatively simple case of a function $f(x, y, z)$ defined on a rectangular box Q in three dimensional space defined by

$$Q = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d \text{ and} r \leq z \leq s\}.$$

We begin by partitioning the region Q by slicing it by planes parallel to the xy-plane, planes parallel to the xz-plane and planes parallel to the yz-plane. Notice that this divides Q into a number of smaller boxes. Number the smaller boxes in any order: $Q_1, Q_2, ..., Q_n$. For each box Q_i $(i = 1, 2, ..., n)$, call the x, y and z dimensions of the box Δx_i , Δy_i and Δz_i , respectively. The volume of the box Q_i is then $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$. As we did in both one and two dimensions, we pick any point (u_i, v_i, w_i) in the box Q_i and form the Riemann sum

$$\sum_{i=1}^{n} f(u_i, v_i, w_i)\Delta V_i.$$

In this three-dimensional case, we define the norm of the partition $\|P\|$

$$\|P\| = \max \{d(Q_1), d(Q_2), \cdots, d(Q_n)\}$$

where $d(Q_n)$ is the diameter of Q_n and d is the Euclid metric in $\mathbb{R}^3$. We can now define the triple integral of $f(x, y, z)$over Q.

Definition 210 *For any function $f(x, y, z)$ defined on the rectangular box Q, we define the triple integral of f over Q by*

$$\iiint\limits_{Q} f(x, y, z)dV = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(u_i, v_i, w_i)\Delta V_i$$

provided the limit exists and is the same for every choice of evaluation points (u_i, v_i, w_i) in Q_i for $i = 1, 2, \cdots, n$. When this happens, we say that f is integrable over Q.

Remark 211 *To evaluate a triple integral we may apply a three dimensional version of Fubini's Theorem.For example suppose that we want to integrate a continuous function $f(x, y, z)$ over a region Q that is bounded below by a surface $z = g(x, y)$, above by another surface $z = h(x, y)$, and on the side by a cylinder C parallel to the $z-$axis.Let R denote the verticle projection of Q on the $xy-$plane. If*

$$R = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq h_1(x)\}$$

then

$$\iiint\limits_Q f(x,y,z)dV = \iint\limits_R \left[\int\limits_{z=g(x,y)}^{z=h(x,y)} f(x,y,z)dz\right] dA$$
$$= \int\limits_a^b \int\limits_{g_1(x)}^{h_1(x)} \int\limits_{g(x,y)}^{h(x,y)} f(x,y,z)dzdydx.$$

In theory of a triple integral can be set up in 6 different orders, but it is up to us to decide which order makes it easiest to evaluate.

8.10.1 Properties of Triple Integrals

We list here three properties of double integrals

1. $\iiint\limits_Q kf(x,y,z)dV = k\iiint\limits_Q f(x,y,z)dV$ for any $k \in \mathbb{R}$.

2. $\iiint\limits_Q [f(x,y,z) \pm g(x,y,z)]\, dV = \iiint\limits_Q f(x,y,z)\, dV \pm \iiint\limits_Q g(x,y,z)\, dV$

3. If $f(x,y,z) \geq 0$ on Q then $\iiint\limits_Q f(x,y,z)\, dV \geq 0$

4. If $f(x,y,z) \geq g(x,y,z)$ on Q then $\iiint\limits_Q f(x,y,z)\, dV \geq \iiint\limits_Q g(x,y,z)\, dA$

Example 212 *We use an integral to compute the volume of the box with opposite corners at* $(0,0,0)$ *and* $(1,2,3)$.

$$\int_0^1 \int_0^2 \int_0^3 dzdydx = \int_0^1 \int_0^2 z|_0^3 dydx = \int_0^1 \int_0^2 3dydx = \int_0^1 3y|_0^2 dx = \int_0^1 6dx = 6.$$

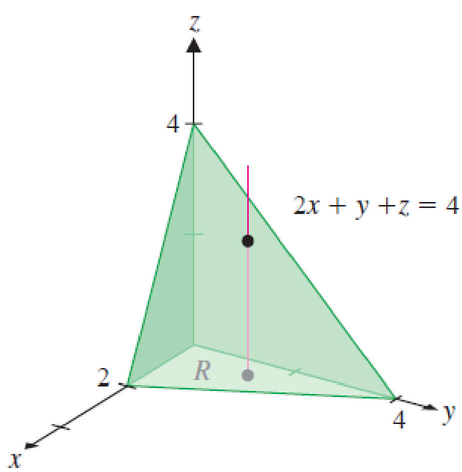
Of course, this is more interesting and useful when the limits are not constant.

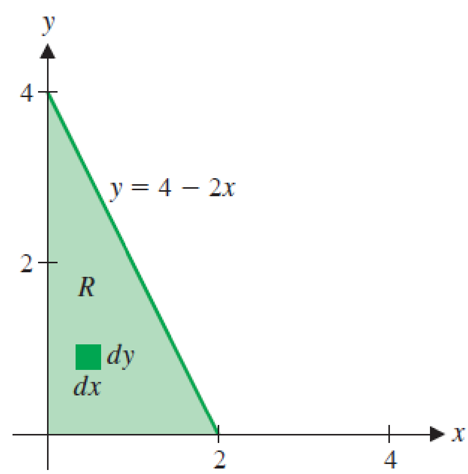
Example 213 *Evaluate* $\int\limits_0^1 \int\limits_0^x \int\limits_0^{x+y} (2x+y-1)\, dzdydx.$

Solution 214

$$\begin{aligned}
\int_0^1\int_0^x\int_0^{x+y}(2x+y-1)\,dz\,dy\,dx &= \int_0^1\int_0^x (2xz+yz+-z)_0^{x+y}\,dydx \\
&= \int_0^1\int_0^x \left(2x^2+3xy+y^2-x-y\right)dydx \\
&= \int_0^1 \left(2x^2y+\frac{3xy^2}{2}+\frac{y^3}{3}-xy-\frac{y^2}{2}\right)_0^x dx \\
&= \int_0^1 (\frac{23x^3}{6}-\frac{3x^2}{2})dx = \left(\frac{23x^4}{24}-\frac{x^3}{2}\right)_0^1 = \frac{11}{24}
\end{aligned}$$

Example 215 *Evaluate* $\iiint\limits_Q 6xydV$*, where* Q *is the tetrahedron bounded by the planes* $x{=}\ 0, y=0, z=0$ *and* $2x+y+z=4$*.*





$$\begin{aligned}
\iiint\limits_Q 6xydV &= \iint\limits_R \int_0^{4-2x-y} 6xydzdydx = \int_0^2\int_0^{4-2x}(6xy)_{z=0}^{z=4-2x-y}\,zdydx \\
&= \int_0^2\int_0^{4-2x} 6xy(4-2x-y)dydx \\
&= \int_0^2 6\left(4x\frac{y^2}{2}-2x^2\frac{y^2}{2}-x\frac{y^3}{3}\right)_{y=0}^{y=4-2x} dx \\
&= \int_0^2 \left[12x\,(4-2x)^2-6x^2\,(4-2x)^2-2x\,(4-2x)^3\right]dx \\
&= \frac{64}{5}
\end{aligned}$$

Example 216 *Q is aregion bounded below by the plane $z = 2$ above by the paraboloid $z = x^2 + y^2 + 4$, and on the side by cylinder $x^2 + y^2 = 1$. Evaluate $\iiint\limits_Q xyzdV$.*

Solution 217

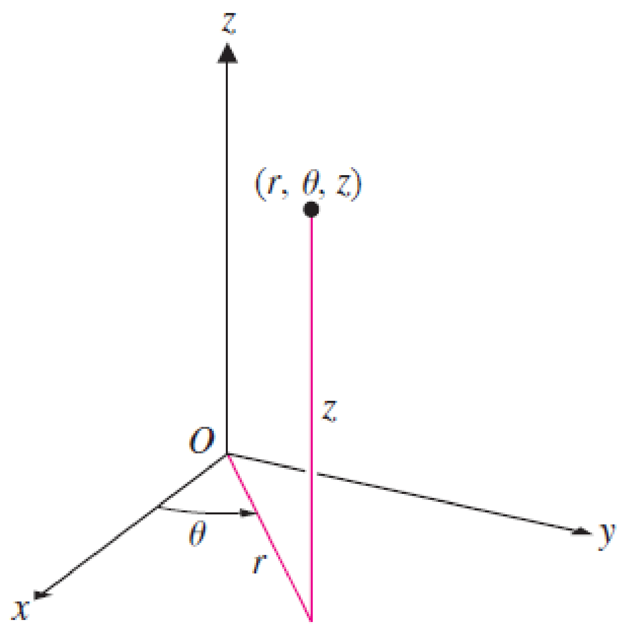
$$\begin{aligned}
I &= \iiint\limits_Q xyzdV = \iint\limits_R \left(\int_2^{x^2+y^2+4} xyzdz \right) dydx \\
&= \iint\limits_R \left(xy\frac{z^2}{2} \right)_2^{x^2+y^2+4} dxdy = \frac{1}{2}\int \int\limits_{x^2+y^2\leq 1} xy \left(\left(x^2+y^2+4\right)^2 - 4 \right) dxdy \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^2 \cos\theta \sin\theta \left(\left(r^2+4\right)^2 - 4 \right) rdrd\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(r^7 + 8r^5 + 12r^3\right) \cos\theta \sin\theta drd\theta \\
&= \frac{107}{48}\int_0^{2\pi} \cos\theta \sin\theta d\theta = \frac{107}{48}\frac{\sin^2\theta}{2}|_0^{2\pi} = 0
\end{aligned}$$

8.10.2 Region Transformations in Triple Integrals

The most important region transformations in triple integrals are done by cylindrical or spherical coordinates. Triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates.

8.10.3 Cylindrical Coordinates

The cylindrical coordinate system is the simplest, since it is just the polar coordinate system plus a z coordinate.

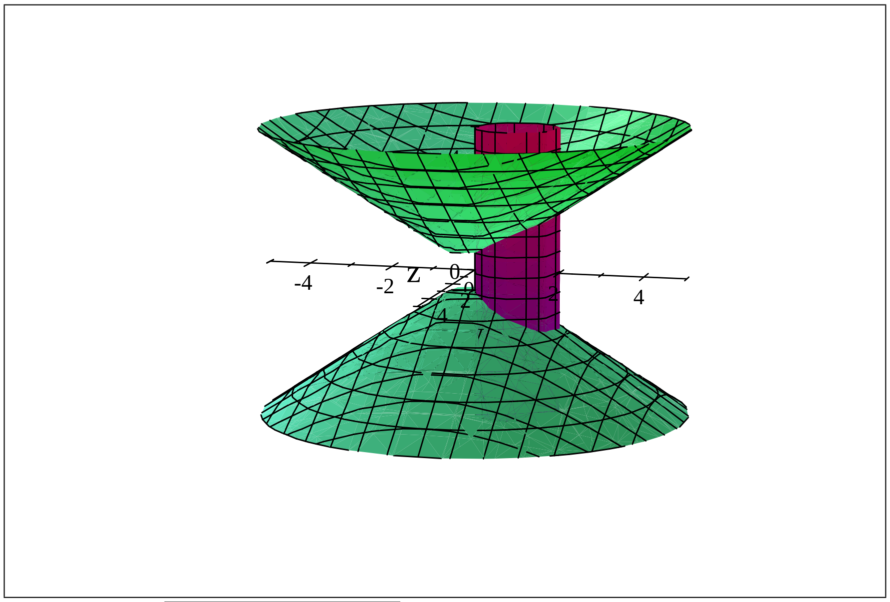


To be precise, we specify a point $P(x,y,z) \in \mathbb{R}^3$ by identifying polar coordinates for the point $(x,y) \in \mathbb{R}^2$: $x = r\cos\theta$ and $y = r\sin\theta$, where$r^2 = x^2 + y^2$ and θ is the angle made by the line segment connecting the origin and the point $(x,y,0)$ with the positive x-axis. Then, $\tan\theta = \frac{y}{x}$. We refer to (r,θ,z) as cylindrical coordinates for the point P.

$$\iiint\limits_{Q} f(x,y,z)dV = \iiint\limits_{Q} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Example 218 *Evaluate* $I = \int\limits_{0}^{2} \int\limits_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int\limits_{0}^{\sqrt{x^2+y^2}} z\left(x^2+y^2\right) dzdxdy.$



$x = r\cos\theta$
$y = r\sin\theta$
$z = z, \ |J| = r$

$z = \sqrt{x^2+y^2} = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = r \Longrightarrow 0 \le z \le r$
$x = \sqrt{2y-y}^2 \Longrightarrow r = 2\sin\theta$

$$\begin{aligned}
I &= \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_0^{\sqrt{x^2+y^2}} z\left(x^2+y^2\right) dzdxdy = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r} zr^2 r dzdrd\theta \\
&= \int_0^{\pi}\int_0^{2\sin\theta} \frac{r^3z^2}{2} |_{z=0}^{z=r} \, dr\theta = \frac{1}{2}\int_0^{\pi}\int_0^{2\sin\theta} r^5 dr\theta \\
&= \frac{1}{12}2^6\int_0^{\pi} \sin^6\theta d\theta = \frac{16}{3}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)^3 d\theta \\
&= \frac{2}{3}\int_0^{\pi}\left(1 - 2\cos 2\theta + \cos^2 2\theta\right)(1-\cos 2\theta)\, d\theta \\
&= \frac{2}{3}\int_0^{\pi}\left(1 - 3\cos 2\theta + 3\cos^2 2\theta - \cos^2 2\theta\cos 2\theta\right) d\theta \\
&= \frac{5\pi}{3}
\end{aligned}$$

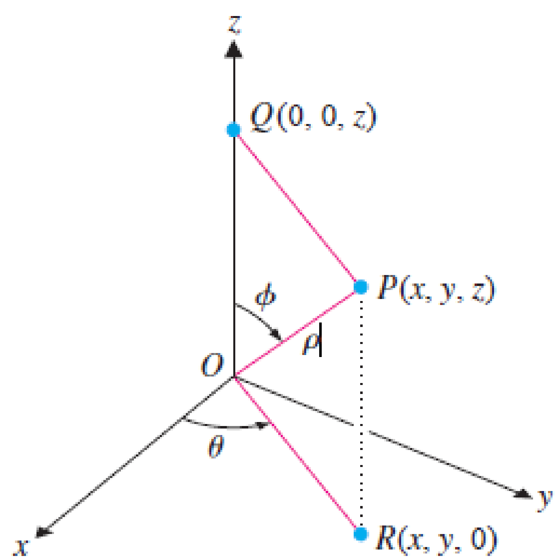
Example 219 *Find the volume under* $z = \sqrt{4-r^2}$ *above the quarter circle inside* $x^2+y^2=4$ *in the first quadrant.*

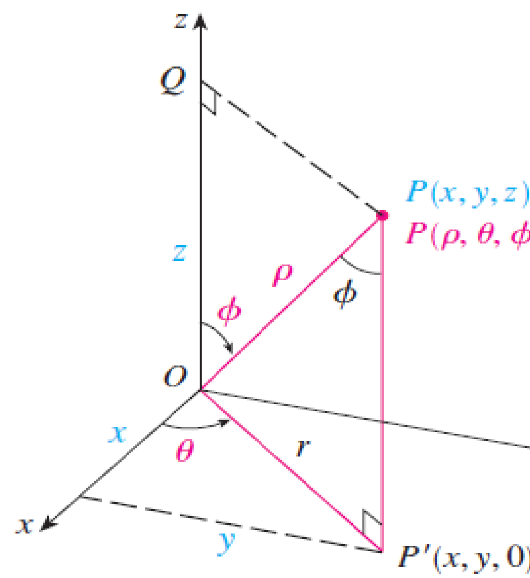
Solution 220 *We could of course do this with a double integral, but we'll use* *a triple integral:*

$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r dz dr d\theta = \int_0^{\pi/2}\int_0^2 \sqrt{4-r^2} r dr d\theta = \frac{4\pi}{3}$$

8.10.4 Spherical Coordinates

The spherical coordinates (ρ, θ, ϕ) of a point P in space are shown in Figure, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.

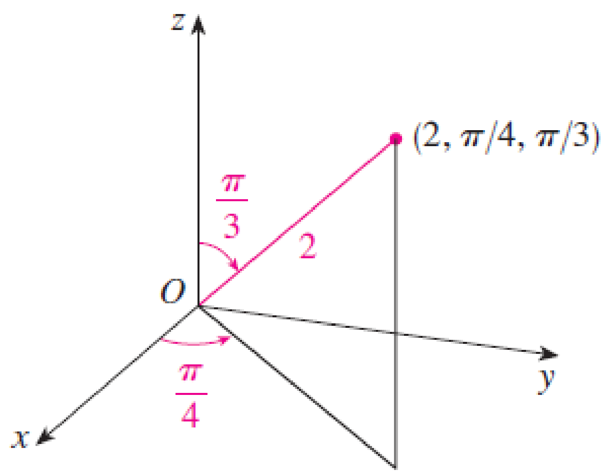




Clearly $|OP| = \rho$, $\cos\phi = \frac{z}{\rho}$, $\sin\phi = \frac{|OP|}{\rho}$, $\cos\theta = \frac{x}{r} = \frac{x}{\rho\sin\phi}$, $\sin\theta = \frac{y}{r} = \frac{y}{\rho\sin\phi}$

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$
$$\rho^2 = x^2 + y^2 + z^2$$

Example 221 *The point* $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ *is given in spherical coordinates. Plot the point and find its rectangular coordinates.*



$$x = \rho \sin\phi \cos\theta = 2 \sin\tfrac{\pi}{3} \cos\tfrac{\pi}{4} = 2\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{2}}{2} = \tfrac{\sqrt{6}}{2}$$
$$y = \rho \sin\phi \sin\theta = 2 \sin\tfrac{\pi}{3} \sin\tfrac{\pi}{4} = 2\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{2}}{2} = \tfrac{\sqrt{6}}{2}$$
$$z = \rho \cos\phi = 2\cos\tfrac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Triple Integrals in Spherical Coordinates We have arrived at the following formula for triple integration in spherical coordinates.

$$\begin{aligned}\iiint_Q f(x,y,z)dV &= \iiint_Q f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,|J|\,d\rho d\theta d\phi \\ &= \iiint_Q f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi\end{aligned}$$

$$|J| = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \rho^2\sin\phi$$

Example 222 *Evaluate* $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2} dzdydx$

Solution 223 *It is usefull to pass spherical coordinates.*

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2}dzdydx$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \frac{1}{1+\rho^2}\rho^2 \sin\phi d\rho d\phi d\theta$$
$$= (4\pi - \pi^2)$$

Exercise 224 $\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}} \frac{(x^2+y^2)^{3/2}}{x^2+y^2+z^2}dzdydx$. *(Answer* $\frac{\pi}{12}$*)*

Exercise 225 $\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}dzdydx$. *(Answer* $\pi\left(1-\frac{\sqrt{2}}{2}\right)$*)*

8.10.5 Applications of Triple Integrals

Volume We can use cyldrical and spherical coordinates to find the volume of the solid.

$$V = \iiint_Q dV = \iiint_Q r dz dr d\theta$$

$$V = \iiint_Q dV = \iiint_Q \rho^2 \sin\phi d\rho d\theta d\phi$$

Example 226 *Find the volume of the sphere of radius a, by using triple integral.*

Solution 227 *To evaluate the volume we will use spherical coordinates.*

$$V = \iiint_{x^2+y^2+z^2\le a^2} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\sin\phi d\rho d\theta d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\sin\phi\ |_0^a\ d\theta d\phi = \int_0^{2\pi}\int_0^{\pi}\frac{a^3}{3}\sin\phi d\theta d\phi$$
$$= \frac{a^3}{3}\int_0^{2\pi} -\cos\phi\ |_0^{\pi}\ d\phi = \frac{2a^3}{3}\phi\ |_0^{2\pi} = \frac{4a^3\pi}{3}\ units^3$$

Example 228 *Find the volume of solid bounded by the paraboloids* $z = 5-x^2-y^2$ *and* $z = 4x^2+4y^2$.