9 LINE INTEGRALS

We have so far integrated "over" intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or "along" a curve—"line integrals" are really "curve integrals".First we are going to investigate line integrals of scalar fields and then line integrals of vector fields.

9.1 Line İntegrals of Scalar Fields

Let assume that $D \subset \mathbb{R}^3,\ f : D \to \mathbb{R}\ u = f(x, y, z)$ is a continuous function and C is a smooth curve in D with the following parametrization form as avector function.

$$C: \quad \overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k};\ a \le t \le b$$

So the function f is continuous on the grapg of C i.e.,

$$P = \{A_0, A_1, \cdots, A_n\} \text{ and } \Delta l_k : \overset{\frown}{A_{k-1}A_k} \text{ (Arc lenght of the curve)}$$

The norm of the partition P defined by

$$\|P\| = \max\{\Delta l_1, \Delta l_2, \cdots, \Delta l_n\}.$$

Let us take a point $M_k(x_k, y_k, z_k)$ on Δl_k $(k = 1, 2, \cdots, n)$ and consider Riemann sum

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta l_k$$

if the limit

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta l_k$$

exists we define

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta l_k = \int_C f(x, y, z)\,dl$$

and it is called first type line integral of f on the curve C. Now we will give how it can be avaluated. Since the point (x, y, z) is on the curve C we can write

$$\begin{aligned} x &= x(t) \\ y &= y(t) \qquad a \le t \le b \\ z &= z(t) \end{aligned}$$

i.e., it satisfies the parametric equation of C.

The arc differential is given by

$$dl = \left\| r'(t) \right\| dt$$

so we write

$$\int_C f(x,y,z)\,dl = \int_{t=a}^{b} f(x(t), y(t), z(t)) \left\| r'(t) \right\| dt$$

Example 238 *Let we consider a curve* C*:*$r(t) = \cos ti + \sin tj + tk,\ 0 \leq t \leq 2\pi$ *and the function* $f(x,y,z) = xyz$*. Then evaluate the following line integral of* f

$$\int_C f(x,y,z)\,dl.$$

Solution 239

$$r'(t) = -\sin ti + \cos tj + k$$

and

$$\begin{aligned}\left\| r'(t) \right\| &= \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2} \\ dl &= \sqrt{2}dt\end{aligned}$$

By using integratation by parts, if $t = u$ *and* $\cos t \sin tdt = dv$ *then* $dt = du$ *and* $v = \frac{\sin^2 t}{2}$.

$$\begin{aligned}\int_C f(x,y,z)\,dl &= \int_C xyzdl = \int_0^{2\pi} (\sin t)(\cos t)(t)\sqrt{2}dt \\ &= \int_0^{2\pi} t \cos t \sin tdt \\ &= \sqrt{2}\left\{ t\frac{\sin^2 t}{2} \mid_0^{2\pi} - \int_0^{2\pi} \frac{\sin^2 t}{2}dt \right\} \\ &= -\frac{\sqrt{2}}{2}\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)dt \\ &= -\frac{\sqrt{2}}{2}\pi\end{aligned}$$

Example 240 *Evaluate* $\int_C (3x + 2y + z)\,dl$ *where* C *is the line segment from* $(0,0,0)$ *to* $(9,3,5)$.

Solution 241 *Lets first find the parametric equation of* C. $\vec{Ax} = \lambda\vec{AB}$, $0 \leq t \leq 1$.

$$(x - a_1, y - a_2, z - a_3) = t(b_1 - a_1, b_2 - a_2, b_3 - a_3)$$

$$\begin{aligned} x &= a_1 + t\,(b_1 - a_1) \\ y &= a_2 + t\,(b_2 - a_1) \\ z &= a_3 + t\,(b_3 - a_1) \end{aligned}$$

$$C: \begin{array}{l} x = 9t \\ y = 3t \\ z = 5t \end{array}, \; 0 \le t \le 1$$

$$\begin{aligned} r(t) &= 9t\vec{i} + 3t\vec{j} + 5t\vec{k}, \quad 0 \le t \le 1 \\ r'(t) &= 9\vec{i} + 3\vec{j} + 5\vec{k} \\ \left\| r'(t) \right\| &= \sqrt{81 + 9 + 25} = \sqrt{115} \\ dl &= \sqrt{115}dt \end{aligned}$$

$$\begin{aligned} \int_C (3x + 2y + z)\,dl &= \int_0^1 (27t + 6t + 5t)\sqrt{115}dt \\ &= 19\sqrt{115} \end{aligned}$$

Remark 242 *If the curve is given on plane we write it as*

$$C : r(t) = x(t)\,\vec{i} + y(t)\,\vec{j},\; a \le t \le b$$

$$\int_C f(x, y)\,dl = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt$$

Example 243 *Compute* $\int_C ye^x ds$ *where* C *is the line segment from* $(1, 2)$ *to* $(4, 7)$.

Solution 244 *We write the line segment as a vector function:* $v = \langle 1, 2\rangle + t\langle 3, 5\rangle$, $0 \le t \le 1$, *or in parametric form* $x = 1 + 3t$, $y = 2 + 5t$. *Then*

$$\int_C ye^x ds = \int_0^1 (2+5t)e^{1+3t}\sqrt{3^2 + 5^2}dt = \frac{16}{9}\sqrt{34}e^4 - \frac{1}{9}\sqrt{34}e.$$

Remark 245 *If the curve* C *is given by polar coordinates, i.e.,* $r = r(\theta)$, $\alpha \le \theta \le \beta$, *then*

$$dl = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}d\theta$$

$$\int_C f(x, y)\,dl = \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}d\theta$$

Example 246 *Evaluate the line integral* $\int\limits_C (x+y)\,dl$, *where* C *is the right above of the Lemniscate,* $r^2 = a^2 \cos 2\theta$.

Solution 247 $r^2 = a^2 \cos 2\theta \Rightarrow r(\theta) = |a| \sqrt{\cos 2\theta} \Rightarrow r^{'}(\theta) = |a| \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}$
$dl = \sqrt{r^2 + (r')^2} d\theta = \frac{|a|}{\sqrt{\cos 2\theta}} d\theta$

$$\begin{aligned} \int\limits_C (x+y)\,dl &= \int\limits_0^{\pi/4} (r\cos\theta + r\sin\theta) \frac{|a|}{\sqrt{\cos 2\theta}} d\theta \\ &= a^2 \end{aligned}$$

Remark 248 *If the equation of curve is given in Cartesian coordinate system like* $y = g(x)$ $a \le x \le b$, *then we write*

$$\int\limits_C f(x,y)\,dl = \int\limits_a^b f(x, g(x)) \sqrt{1 + (g'(x))^2} dx$$

Example 249 *Evaluate the line integral* $\int\limits_C x dl$, *where* $C:\ y = \frac{x^2}{2},\ 0 \le x \le 1.$

Solution 250 $g'(x) = x \Rightarrow dl = \sqrt{1+x^2} dx$

$$\int\limits_C x dl = \int\limits_0^1 x\sqrt{1+x^2} dx = \frac{1}{3} 2^{3/2} - \frac{1}{3}$$

The Properties of First Type Line Integrals

Let assume that f and g are integrable on the curve C

1. $\int\limits_C (f+g)\,dl = \int\limits_C f dl + \int\limits_C g dl$

2. $\int\limits_C \alpha f dl = \alpha \int\limits_C f dl,\ \alpha \in \mathbb{R}$

3. If $f(x,y,z) \ge 0$, then $\int\limits_C f(x,y,z)\,dl \ge 0$.

4. If f is integrable on the curve C, then $|f|$ is also integrable and

$$\left| \int\limits_C f(x,y,z)\,dl \right| \le \int\limits_C |f(x,y,z)|\,dl$$

5. If the curves C_1 and C_2 are contigous and $C = C_1 \cup C_2$, then

$$\int_C f dl = \int_{C_1} f dl + \int_{C_2} f dl$$

Example 251 *Evaluate the integral* $\int_C (2x + y + 3z)\, dl$, *where* C *is the broken line which associates the points* $O(0,0,0)$, $A(1,1,0)$ *and* $B(1,1,1)$.

Solution 252 *By using the property* (5)

$$\int_C (2x + y + 3z)\, dl = \int_{C_1} (2x + y + 3z)\, dl + \int_{C_2} (2x + y + 3z)\, dl$$

$$C_1 : \begin{array}{c} x(t) = t \\ y(t) = t \\ z(t) = 0 \end{array} \Rightarrow C_1(t) = ti + tj$$

$$C_2 : \begin{array}{c} x(t) = 1 \\ y(t) = 1 \\ z(t) = t \end{array} \Rightarrow C_1(t) = i + j + tk$$

where $0 \leq t \leq 1$.

$$d_1 l = \left\| C_1'(t) \right\| dt = \sqrt{2} dt$$

$$d_2 l = \left\| C_2'(t) \right\| dt = dt$$

$$\begin{aligned} \int_C (2x + y + 3z)\, dl &= \int_0^1 (2t + t)\sqrt{2} dt + \int_0^1 (2 + 1 + 3t)\, dt \\ &= \frac{9 + 3\sqrt{2}}{2} \end{aligned}$$

9.2 Line Integrals Of Vector Fields

In this section we will evaluate line integrals of vector fields.

$$F(x,y,z) = P(x,y,z)\, i + Q(x,y,z)\, j + R(x,y,z)\, k$$

$$r(t) = x(t)\, i + y(t)\, j + z(t)\, k, \ a \leq t \leq b$$

Let assume that $A = (x(a), y(a), z(a))$ is the beginning point $A = (x(b), y(b), z(b))$ is the end point and $P = \{A_0, A_1, \cdots, A_n\}$ is apartition of the curve. Consider that a particle on this curve moves from the point A_{k-1} to the point A_k with the effect of the force F in a time period of $\Delta t = t_k - t_{k-1}$. So the labor or workforce is approximately given by

$$W \cong F \overset{\frown}{A_{k-1}A_k} \cong F\Delta r_k$$

$$\Delta r_k = T_k \Delta l_k$$

The point (x_k, y_k, z_k) is on arch $\overset{\frown}{A_{k-1}A_k}$. The lenght of Arc $\Delta l_k := \overset{\frown}{A_{k-1}A_k}$

$$W \cong \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta r_k$$

The labour from A to B is given by

$$\begin{aligned} W &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta r_k = \int_C F(x,y,z)\,dr \\ &= \int_C F(x,y,z)\cdot T(x,y,z)\,dl \end{aligned}$$

where T is the unit tangent vector on the point A_{k-1} and is given by

$$T(t) = \frac{r'(t)}{\|r'(t)\|}.$$

Definition 253 *Let assume that C is a smooth curve and F is a vector field on C, T is the unit tangent vector of the curve on the point $(x(t), y(t), z(t))$ with the same direction of C. Then the integral*

$$\int_C F\cdot dr = \int_C F\cdot T dl$$

is called second type line integral of vector field F on the curve C.

$C: r(t) = x(t)\,i + y(t)\,j + z(t)\,k,\ a \le t \le b$
$T(x(t), y(t), z(t)) = \frac{r'(t)}{\|r'(t)\|},\quad dl = \left\|r'(t)\right\| dt$

$$\begin{aligned} \int_C F(x,y,z)\cdot T(x,y,z)\,dl &= \int_a^b F(x(t), y(t), z(t))\,\frac{r'(t)}{\|r'(t)\|}\left\|r'(t)\right\| dt \\ &= \int_a^b \left[F(x(t), y(t), z(t))\,\frac{dr}{dt}\right] dt \end{aligned}$$

Example 254 *Evaluate* $\int\limits_C F\cdot dr$ *where* $F(x,y,z)=\left(3x^2-3y\right)i+3zj+(xyz)\,k$ *and* C *is the curve segment from* $O(0,0,0)$ *to* $A(1,1,1)$ *and give by* $r(t)=ti+tj+tk,\ 0\le t\le 1.$

Solution 255 $\frac{dr}{dt}=i+j+k$

$$\begin{aligned}\int\limits_C F\cdot dr &= \int\limits_0^1 \left[\left(3t^2-3t\right)i+3tj+t^3k\right]\left[i+j+k\right]dt\\ &= \int\limits_0^1 \left(3t^2-3t+3t+t^3\right)dt=\frac{5}{4}\end{aligned}$$

Remark 256 $\int\limits_{-C} F\cdot dr=-\int\limits_C F\cdot dr$

Remark 257 *Given the vector field* $F(x,y,z)=P(x,y,z)\,i+Q(x,y,z)\,j+R(x,y,z)\,k$ *and the smooth curve* C *parametrized by*

$$C:r(t)=x(t)\,i+y(t)\,j+z(t)\,k,\ \ a\le t\le b$$

$$\begin{aligned}\int\limits_C F\cdot dr &= \int\limits_C F(x,y,z)\left(\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k\right)dt\\ &= \int\limits_C \left(P(x,y,z)\,i+Q(x,y,z)\,j+R(x,y,z)\,k\right)\left(\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k\right)dt\\ &= \int\limits_C P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz\\ &= \int\limits_a^b \left\{\begin{array}{c} P(x(t),y(t),z(t))\,x'(t)+Q(x(t),y(t),z(t))\,y'(t)\\ +R(x(t),y(t),z(t))\,z'(t)\end{array}\right\}dt\end{aligned}$$

$\int\limits_C P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz$ is the second type line lintegral of vector field

Example 258 *Evaluate* $\int\limits_C xydx+xzdy+yzdz,$ *where* C *is defined by*

$$C:r(t)=t^2i+tj+t^3k,\ 0\le t\le 1$$

Solution 259 $\int\limits_C xydx + xzdy + yzdz$ *is a second type line integral of vector field.2*

$$\begin{aligned}\int\limits_C xydx + xzdy + yzdz &= \int\limits_0^1 \left(t^2 t2t + t^2 t^3 + tt^3 3t^2\right) dt \\ &= \int\limits_0^1 \left(2t^4 + t^5 + 3t^6\right) dt \\ & \frac{209}{210}\end{aligned}$$

9.3 Fundamental Properties of Line Integrals

Theorem 260 *Let f be a continuously differentiable function on C, where C is a smooth curve which starts at (x_1, y_1, z_1) and ends at (x_2, y_2, z_2). Then*

$$\begin{aligned}\int\limits_C \operatorname{grad} f \cdot dr &= \int\limits_C \nabla f \cdot dr \\ &= f(x_2, y_2, z_2) - f(x_1, y_1, z_1)\end{aligned}$$

where $\operatorname{grad} f = \nabla f = f_x i + f_y j + f_z k$.

Example 261 *Evaluate* $\int\limits_C F dr$ *where*

$$C : r(t) = \cos t i + \sin t j + t k,\ 0 \le t \le 2\pi$$

and

$$F(x, y, z) = 2xy^3 z i + 3x^2 y^2 z j + x^2 y^3 k.$$

Solution 262 *First way:*

$$\begin{aligned}\int\limits_C F dr &= \int\limits_C \left(2xy^3 z\right) dx + (3x^2 y^2 z) dy + x^2 y^3 dz \\ &= \int\limits_0^{2\pi} 2\cos t (\sin^3 t) t (-\sin t) dt \\ &\quad + \int\limits_0^{2\pi} 3\cos^2 t \left(\sin^2 t\right) t (\cos t)\, dt \\ &\quad + \int\limits_0^{2\pi} \cos^2 t \left(\sin^3 t\right) dt \\ &= 0\end{aligned}$$