Solution 259 $\int\limits_C xydx + xzdy + yzdz$ *is a second type line integral of vector field.2*

$$\begin{aligned}\int\limits_C xydx + xzdy + yzdz &= \int\limits_0^1 \left(t^2t2t + t^2t^3 + tt^33t^2\right)dt \\ &= \int\limits_0^1 \left(2t^4 + t^5 + 3t^6\right)dt \\ &\quad \frac{209}{210}\end{aligned}$$

9.3 Fundamental Properties of Line Integrals

Theorem 260 *Let f be a continuously differentiable function on C, where C is a smooth curve which starts at (x_1, y_1, z_1) and ends at (x_2, y_2, z_2). Then*

$$\begin{aligned}\int\limits_C \operatorname{grad} f \cdot dr &= \int\limits_C \nabla f \cdot dr \\ &= f(x_2, y_2, z_2) - f(x_1, y_1, z_1)\end{aligned}$$

where $\operatorname{grad} f = \nabla f = f_x i + f_y j + f_z k.$

Example 261 *Evaluate* $\int\limits_C F dr$ *where*

$$C : r(t) = \cos t i + \sin t j + t k,\ 0 \le t \le 2\pi$$

and

$$F(x, y, z) = 2xy^3zi + 3x^2y^2zj + x^2y^3k.$$

Solution 262 *First way:*

$$\begin{aligned}\int\limits_C F dr &= \int\limits_C \left(2xy^3z\right)dx + (3x^2y^2z)dy + x^2y^3dz \\ &= \int\limits_0^{2\pi} 2\cos t(\sin^3 t)t(-\sin t)dt \\ &\quad + \int\limits_0^{2\pi} 3\cos^2 t\left(\sin^2 t\right)t\left(\cos t\right)dt \\ &\quad + \int\limits_0^{2\pi} \cos^2 t\left(\sin^3 t\right)dt \\ &= 0\end{aligned}$$

Second Way:If we assume $f(x,y,z) = x^2y^3z$ *then* $\operatorname{grad} f(x,,y,z) = F(x,y,z)$

$$\begin{aligned}\int_C F dr &= \int_C \operatorname{grad} f \cdot dr \\ &= f(1,0,2\pi) - f(1,0,0) = 0 - 0 = 0\end{aligned}$$

Definition 263 *First suppose that* $\overrightarrow{F}$ *is a continuous vector field in some domain* D.

1. $\overrightarrow{F}$ *is conservative vector field if there is a function* f *such that* $\overrightarrow{F} = \nabla f = \operatorname{grad} f$. *The function* f *is called a potential function for the vector field.*
2. $\int_C F dr$ *is independent of path if* $\int_{C_1} F dr = \int_{C_2} F dr$ *for any two paths* C_1 *and* C_2 *in* D *with the same initial and final points.*
3. *A path* C *is called closed if its initial and end points are the same point. For example a circle is aclosed path.*
4. *A path* C *is simple if it doesn't cross itself. A circle is a simple curve.*
5. *A region* D *is open if it doesn't any of its boundary point.*
6. *A region* D *is connected if we can connect any two points in the region with a path that lies completly in* D.

Theorem 264 *If the vector field* F *is a gradient vector of a differentiable function* f *then* $\int_C F \cdot dr$ *is independent of path.*

Notation 265 *Let assume the vector field as*

$$F(x,y,z) = Pi + Qj + Rk.$$

If $F = \operatorname{grad} f$ *i.e.,*$f_x = P,\ f_y = Q,\ f_z = R$ *then we can write* $f_x dx + f_y dy + f_z dz = df$. *So* $Pdx + Qdy + Rdz$ *is an exact differential.*

Remark 266 *Let the functions* $P,\ Q,\ R$ *are continuous in a region* D. *If* $Pdx+Qdy+Rdz$ *is an exact differential then for any two points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *in* D *and any curve* C *in* D *from* (x_1, y_1, z_1) *to* (x_2, y_2, z_2), *then*

$$\int_C Pdx + Qdy + Rdz = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Theorem 267 *If C is a simple closed curve in D and $\int_C F dr$ is independent of path in D, then*

$$\int_C F \cdot dr = 0$$

Notation 268 *We can use the symbol $\oint_C$ to show that the curve is simple and closed.*

Notation 269 *If $df = Pdx + Qdy$ satisfies then $f_x = P$ and $f_y = Q$*

$$f_x(x,y) = P(x,y) \tag{10}$$
$$f_y(x,y) = Q(x,y) \tag{11}$$

*By using (**??**) we write*

$$f(x,y) = \int P(x,y)\,dx + \varphi(y) \tag{12}$$

Then if we differentiate the equation (12) with respect to y we write

$$Q(x,y) = f_y(x,y) = \frac{d}{dy}\int P(x,y)\,dx + \varphi'(y)$$

*by using (**??**). Using the last equation by taking integral we get $\varphi(y)$. Finally we obtain the function f. Then if the beginning point of curveis (x_1, y_1) and end point (x_2, y_2) we can write integral as;*

$$\int_C Pdx + Qdy = \int_{(x_1,y_1)}^{(x_2,y_2)} Pdx + Qdy = f(x_2,y_2) - f(x_1,y_1).$$

Example 270 *Evaluate the integral $\int_{(0,0)}^{(1,2)} \left(y^2 - 2xy\right)dx + \left(2xy - x^2 + 1\right)dy$.*

Solution 271 $P(x,y) = \left(y^2 - 2xy\right)$ *and* $Q(x,y) = \left(2xy - x^2 + 1\right)$. *Since*

$$P_y = 2y - 2x = Qx$$

$Pdx + Qdy$ is an exact differential. So the integral is independent of path. Now we will find the function f such that

$$df = Pdx + Qdy.$$

$$f_x = P(x,y) = \left(y^2 - 2xy\right)$$
$$f_y = Q(x,y) = \left(2xy - x^2 + 1\right)$$

$$\begin{aligned} f(x,y) &= \int \left(y^2 - 2xy\right) dx + \varphi(y) \\ &= \left(y^2 x - x^2 y\right) dx + \varphi(y) \end{aligned}$$

$$2yx - x^2 + \varphi^{'}(y) = f_y = Q(x,y) = \left(2xy - x^2 + 1\right)$$
$$\varphi^{'}(y) = 1 \Rightarrow \varphi(y) = y$$

Finally

$$f(x,y) = y^2 x - x^2 y + y$$

$$\begin{aligned} \int\limits_{(0,0)}^{(1,2)} \left(y^2 - 2xy\right) dx + \left(2xy - x^2 + 1\right) dy &= f(1,2) - f(0,0) \\ &= 4 \end{aligned}$$

Remark 272 *If D is asimple connected regioan and $P,\ Q,\ R$ are the functions with continuous derivatives in D, then $Pdx + Qdy + Rdz$ is an exact differential if and only if*

$$P_y = Q_x,\ P_z = R_x,\ Q_z = R_y$$

Example 273 *Evaluate the integral*

$$I = \int\limits_{(0,0,0)}^{(1,1,1)} (yz + y + z + 1)\, dx + (xz + x + z + 2)\, dy + (xy + x + y + 3)\, dz.$$

Solution 274 $P(x,y,z) = (yz + y + z + 1)$, $Q(x,y,z) = (xz + x + z + 2)$ *and* $R(x,y,z) = (xy + x + y + 3)$. *Since*

$$P_y = Q_x,\ P_z = R_x,\ Q_z = R_y$$

$Pdx + Qdy + Rdz$ is an exact differential.

$$f(x,y,z) = xyz + xy + xz + x + zy + 2y + 3z$$

$$\begin{aligned} I &= \int\limits_{(0,0,0)}^{(1,1,1)} (yz + y + z + 1)\, dx + (xz + x + z + 2)\, dy + (xy + x + y + 3)\, dz \\ &= f(1,1,1) - f(0,0,0) = 10 \end{aligned}$$

Theorem 275 (*Green's Theorem*) *Let B be a closed plane region bounded by a simple curve C.If the vector field $F(x,y) = P(x,y)\,i + Q(x,y)\,j$ is continuously differentiable (i.e., P and Q have continuous first partial derivatives) if C is traversed in the counter-clockwise direction then*

$$\oint_C F \cdot dr = \int_C P(x,y)\,dx + Q(x,y)\,dy$$
$$= \iint_B \left(\frac{\partial Q}{\partial x} \frac{-\partial P}{\partial y} \right) dxdy$$

Example 276 *Evaluate $I = \oint_C 2xy^3 dx + 4x^2y^2 dy$, where C is the boundary of the region B bounded by the curve $y = x^3$, the line $x = 1$ and x-axis.(C is directed in the possitive direction)*

Solution 277 $P(x,y) = 2xy^3$ *and* $Q(x,y) = 4x^2y^2$. $\frac{\partial Q}{\partial x} \frac{-\partial P}{\partial y} = 8xy^2 - 6xy^2 = 2xy^2$

$$I = \oint_C 2xy^3 dx + 4x^2y^2 dy == \iint_B \left(\frac{\partial Q}{\partial x} \frac{-\partial P}{\partial y} \right) dxdy$$
$$= \iint_B 2xy^2 dxdy = \int_0^1 \int_0^{x^3} 2xy^2 dxdy = \frac{2}{33}$$

9.4 Applications of Line Integrals

9.4.1 Area

Let assume that B is a simple region in $xy-$ plane and C is a boundary curve of B which is traversed in the counter clockwise direction. Then by using Green's Formula we get the area of region B bounded by closed curve C as

$$A = \int_C xdy = \iint_B dxdy$$

$$A = \int_C -ydx = \iint_B dxdy$$

If we sum these equations we find

$$A = \frac{1}{2} \int_C xdy - ydx$$