Theorem 275 (*Green's Theorem*) *Let B be a closed plane region bounded by a simple curve C.If the vector field $F(x,y) = P(x,y)\,i + Q(x,y)\,j$ is continuously differentiable (i.e., P and Q have continuous first partial derivatives) if C is traversed in the counter-clockwise direction then*

$$\begin{aligned}\oint_C F \cdot dr &= \int_C P(x,y)\,dx + Q(x,y)\,dy \\ &= \iint_B \left(\frac{\partial Q}{\partial x} \frac{-\partial P}{\partial y} \right) dxdy\end{aligned}$$

Example 276 *Evaluate $I = \oint_C 2xy^3 dx + 4x^2y^2 dy$, where C is the boundary of the region B bounded by the curve $y = x^3$, the line $x = 1$ and x-axis.(C is directed in the possitive direction)*

Solution 277 $P(x,y) = 2xy^3$ *and* $Q(x,y) = 4x^2y^2$. $\frac{\partial Q}{\partial x} \frac{-\partial P}{\partial y} = 8xy^2 - 6xy^2 = 2xy^2$

$$\begin{aligned}I &= \oint_C 2xy^3 dx + 4x^2y^2 dy == \iint_B \left(\frac{\partial Q}{\partial x} \frac{-\partial P}{\partial y} \right) dxdy \\ &= \iint_B 2xy^2 dxdy = \int_0^1 \int_0^{x^3} 2xy^2 dxdy = \frac{2}{33}\end{aligned}$$

9.4 Applications of Line Integrals

9.4.1 Area

Let assume that B is a simple region in $xy-$ plane and C is a boundary curve of B which is traversed in the counter clockwise direction. Then by using Green's Formula we get the area of region B bounded by closed curve C as

$$A = \int_C xdy = \iint_B dxdy$$

$$A = \int_C -ydx = \iint_B dxdy$$

If we sum these equations we find

$$A = \frac{1}{2}\int_C xdy - ydx$$

Example 278 *Find the area of a circle with radius a by using line integrals.*

Solution 279 *Lets first write the parametric equation of circle.*

$$C : r(t) = a\cos t i + a\sin t j,\ 0 \leq t \leq 2\pi$$

$$\begin{aligned} A &= \frac{1}{2}\int\limits_C xdy - ydx = \frac{1}{2}\int\limits_0^{2\pi} \left[(a\cos t)(a\cos t) + (a\sin t)(a\sin t)\right] dt \\ &= \frac{1}{2}\int\limits_0^{2\pi} a^2 dt = \pi a^2 \ units^2 \end{aligned}$$

9.4.2 Lenght of a Curve

If C is a smooth curve and f is an integrable function defined on C, then the length of the curve is

$$L = \int\limits_C dl.$$

Example 280 *Find the arclength of the following curve.*

$$r(t) = ti + \frac{3}{2}t^2 j + \frac{3}{2}t^3 k,\ 0 \leq t \leq 2$$

Solution 281

$$\begin{aligned} L &= \int\limits_C dl = \int\limits_0^2 \left\| r'(t) \right\| dt \\ &= \int\limits_0^2 \sqrt{1 + (3t)^2 + \left(\frac{9}{2}t^2\right)^2} dt \\ &= \int\limits_0^2 \sqrt{\left(1 + \left(\frac{9}{2}t^2\right)\right)^2} dt = \int\limits_0^2 \left(1 + \left(\frac{9}{2}t^2\right)\right) dt \\ &= \left(t + \frac{3}{2}t^3\right)\Big|_0^2 = 14 \ units \end{aligned}$$

9.4.3 Mass and the center of a Mass

Let assume a wire on the curve C. If the density of this wire on the point (x, y, z) $\sigma(x, y, z)$ is continuous then the mass is

$$M = \int\limits_C \sigma(x, y, z)\, dl.$$

The center of a mass of a wire given by

$$\begin{aligned}\bar{x} &= \frac{1}{M}\int_C x\sigma(x,y,z)\,dl\\ \bar{y} &= \frac{1}{M}\int_C y\sigma(x,y,z)\,dl\\ \bar{z} &= \frac{1}{M}\int_C z\sigma(x,y,z)\,dl\end{aligned}$$

Example 282 *Let assume a wire ring on a circle $x^2+y^2=a^2$ on $xy-$plane. If the density of this ring on the point (x,y) is $\sigma(x,y)=|x|+\frac{a}{\pi}$, then evaluate the mass and the coordinates of the center of mass of this ring.*

Solution 283 *Since the wire on $xy-$plane $\bar{z}=0$. The parametric equation of circle is*

$$r(t)=a\cos ti+a\sin tj,\ \ 0\le t\le 2\pi$$

$$\begin{aligned}M &= \int_C \sigma(x,y,z)\,dl=\int_C\left(|x|+\frac{a}{\pi}\right)dl=\int_0^{2\pi}\left(|a\cos t|+\frac{a}{\pi}\right)adt\\ &= a\left\{\begin{array}{l}\int_0^{\pi/2}\left(a\cos t+\frac{a}{\pi}\right)dt+\int_{\pi/2}^{\pi}\left(-a\cos t+\frac{a}{\pi}\right)dt\\ +\int_{\pi}^{3\pi/2}\left(-a\cos t+\frac{a}{\pi}\right)dt+\int_{3\pi/2}^{2\pi}\left(a\cos t+\frac{a}{\pi}\right)dt\end{array}\right\}\\ &= 6a^2\end{aligned}$$

$$\begin{aligned}\bar{x} &= \frac{1}{M}\int_C x\sigma(x,y,z)\,dl=\frac{1}{6a^2}\int_0^{2\pi}a^2\cos t\left(|a\cos t|+\frac{a}{\pi}\right)dt\\ &= 0\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{1}{M}\int_C y\sigma(x,y,z)\,dl=\frac{1}{6a^2}\int_0^{2\pi}a^2\sin t\left(|a\cos t|+\frac{a}{\pi}\right)dt\\ &= 0\end{aligned}$$

9.4.4 Calculation of Work

Work done by a force F on an object moving along a curve C is described by the line integral

$$W = \int_C F \cdot dr$$

where dr is the unit tangent vector. If the force F given by

$$F(x, y, z) = P(x, y, z)\, i + Q(x, y, z)\, j + R(x, y, z)\, k$$

then the formula of work is given by

$$W = \int_C F \cdot dr = \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

Example 284 *A particle is moving upwards under the influence of the force F,*

$$F(x, y, z) = -yzi + xzj + xyk$$

on the Helix curve

$$r(t) = \cos ti + \sin tj + tk.$$

Evaluate the work.

Solution 285

$$\begin{aligned} W &= \int_0^{2\pi} -yzdx + xzdy + xydz \\ &= \int_0^{2\pi} [(-t\sin t(-\sin t)) + t\cos t(\cos t) + \cos t\sin t]\, dt \\ &= \int_0^{2\pi} (t + \cos t\sin t)\, dt \\ &= 2\pi^2 \end{aligned}$$