10.3 Fundamental Theorems of Surface Integrals

In this section we will give Stoke's Theorem and Divergence Theorem of surface integrals. Actually these theorems are generalizations of Greenn Theorem to the three dimensional spaces.

Theorem 292 (Stoke's Theorem) *Let S be an directed surface with boundary C which is piecewise smooth, and let*

$$F(x,y,z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$$

be a vector field defined on a region containing S. If P, Q, R have continuous partial derivatives, then

$$\oint_C F \cdot dr = \iint_S (rotF) \cdot nds$$

where n is the normal of S.

Notation 293 *Let a vector field $F(x,y,z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$ be such that P, Q, R have continuous partial derivatives in same region. The* curl *(rot) of F is given by*

$$\begin{aligned} \nabla \times F &= \operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\,i + (P_z - R_x)\,j + (Q_x - P_y)\,k \end{aligned}$$

Example 294 *Assume that $F(x,y,z) = 4yi + xj + 2zk$, S be the above part of the sphere $x^2 + y^2 + z^2 = a^2$. The normal vector n is directed tot he outside of the sphere. Evaluate $\iint_S (\operatorname{curl} F) \cdot nds$.*

Solution 295 $z = f(x,y) = \sqrt{a^2 - x^2 - y^2}$

$$\begin{aligned} C &: r(t) = a\cos t i + a\sin t j,\ 0 \le t \le 2\pi \\ r'(t) &= -a\sin t i + a\cos t j, \end{aligned}$$

$$\begin{aligned}
\iint_S (\operatorname{curl} F)\cdot n ds &= \oint_C F\cdot dr \\
&= \int_0^{2\pi} (4a\sin t i + a\cos t j + 0k)(-a\sin t i + a\cos t j)\, dt \\
&= \int_0^{2\pi} \left(-4a^2\sin^2 t + a^2\cos^2 t\right) dt = \int_0^{2\pi} \left(-4a^2\sin^2 t + a^2\left(1-\sin^2 t\right)\right) dt \\
&= \int_0^{2\pi} \left(-5a^2\sin^2 t + a^2\right) \\
&= \frac{-5}{2}a^2 2\pi + a^2 2\pi = -3a^2\pi
\end{aligned}$$

Example 296 *Let $F(x,y,z) = e^x \sin y i + (e^x\cos y - z) j + yk$, S be the part of the cone $z = 1-\sqrt{x^2+y^2}$ which remains on the top of the $xy-$plane and n normal vector is oriented to the positive direction. Then evaluate the integral of F over the boundary curve of S. $\left(\int_{\partial S} F\cdot dr = ?\right)$*

Solution 297 *It is diffucult to evaluate this integral by classical way like the previous example. So we wil use Stokes Theorem.*

$$\begin{aligned}
\operatorname{curl} F &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= (R_y - Q_z) i + (P_z - R_x) j + (Q_x - P_y) k \\
&= (1+1) i + (0-0) j + (e^x\cos y - e^x \cos y) = 2i
\end{aligned}$$

$$n = \frac{-f_x i - f_y j + k}{\sqrt{1+f_x^2+f_y^2}} = \frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}} i + \frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}} j + \frac{1}{\sqrt{2}} k$$

$$\begin{aligned}
\int_{\partial S} F\cdot dr &= \iint_S \operatorname{curl} F\cdot n ds \\
&= \iint_S 2i\left(\frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}} i + \frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}} j + \frac{1}{\sqrt{2}} k\right) ds \\
&= \iint_B \frac{2x}{\sqrt{2}\left(x^2+y^2\right)} \sqrt{2} dx dy \\
&= \int_0^{2\pi}\int_0^1 \frac{2r\cos\theta}{r} r dr d\theta = \int_0^{2\pi} \cos\theta d\theta = \sin\theta|_0^{2\pi} = 0
\end{aligned}$$

Definition 298 *Let D is a region in $xyz-$space. If the region D is between the graphs of functions f_1, f_2 defined in a region B_1 on xy-plane, between the graphs of functions $g_{1,}$ g_2 defined in a region in B_2 on xz-plane and between the graphs of functions h_1, h_2 defined in a region B_3 on yz-plane. Then D is called a simple solid region.*

Let D be a simple solid region, S is the boundary surface of D with positive orientation and n is the normal of this space with the direction outward from D. Let

$$F(x, y, z) = P(x, y, z)\, i + Q(x, y, z)\, j + R(x, y, z)\, k$$

be a vector field whose components have continuous first order partial derivatives on D. Then

$$\iint\limits_S F \cdot n ds = \iiint\limits_D \operatorname{div} F dv$$

where $\operatorname{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Example 299 *Find the integral of vector field $F = xi + 2yj + 3zk$ over the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$*

Solution 300

$\operatorname{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1 + 2 + 3 = 6$

$$\begin{aligned} \iint\limits_S F \cdot n ds &= \iiint\limits_D \operatorname{div} F dv \\ &= 6 \iiint\limits_D dxdydz \\ &= 6 \frac{4}{3} \pi abc \end{aligned}$$

10.4 Applications of Surface Integrals

10.4.1 Calculation of Surface Area

The area of surface S is given by

$$A = \iint\limits_S ds$$

where $ds = \sqrt{1 + (f_x)^2 + (f_y)^2} dxdy$ and $z = f(x, y)$

Exercise 301 *Calculate the surface area of the sphere with radius a.*

10.4.2 Calculation of Mass and Center of Mass

Let assume that the piece of a surface S given as a metal plate and suppose that the density per unit area of the surface is given by the function $\sigma(x, y, z)$ .Then we can write the mass of sheet and center of mass by

$$M = \iint_S \sigma(x, y, z)\, ds$$

$$\begin{aligned}
\bar{x} &= \frac{1}{M} \iint_S x\sigma(x, y, z)\, ds \\
\bar{y} &= \frac{1}{M} \iint_S y\sigma(x, y, z)\, ds \\
\bar{z} &= \frac{1}{M} \iint_S z\sigma(x, y, z)\, ds
\end{aligned}$$

$$G = (\bar{x}, \bar{y}, \bar{z})$$