MTH3338 PARTIAL DIFFERENTIAL EQUATIONS

The books we will use in this course are given as follows:

1. Ian Sneddon , Elements of Partial Differential Equations, McGraw-Hill International Editions (Mathematics Series), 1985

2. Richard Haberman, Applied Partial Differential Equations: with Fourier Series and Boundary Value Problems (Fourth Edition), Pearson Education (2004)

SECTION 1. ORDINARY DIFFERENTIAL EQUATIONS IN MORE THAN TWO VARIABLES

1.1. Curves and Surfaces in 3-dimensional space

Surfaces in Three Dimensions

If the rectangular cartesian coordinates (x, y, z) of a point in three dimensional space are connected by a single relation of the type

$$f(x, y, z) = 0 \tag{1}$$

the point lies on a surface. For this reason, we call the relation (1) the equation of a surface S. In other words, equation (1) is a relation satisfied by points which lie on a surface.

Such a surface is also represented by the equation $z = F(x, y)$.

In three dimensional space, there is another important representation of the surfaces. If we have a set of relations of the form

$$x = F_1(u, v) \ , \ y = F_2(u, v) \, , \ z = F_3(u, v) \tag{2}$$

then to each pair of values of u,v there corresponds a set of numbers (x, y, z) and hence a point in space.

If we solve the first pair of equations

$$x = F_1(u, v) \ , \ y = F_2(u, v) \, ,$$

we can write u and v as functions of x and y

$$u = \lambda(x, y) \ , \ v = \mu(x, y) \, .$$

The corresponding value of z is obtained by substituting these values for u and v into the third of the equation (2). That is, the value of z is determined as

$$z = F_3(\lambda(x, y), \mu(x, y))$$

so that there is a functional relation of type (1) between the three coordinates x, y and z. Equation (1) expresses that the point (x, y, z) lies on a surface. The equations (2) express that any point (x, y, z) determined from them always lies on a fixed surface. For this reason, equations of this type are called **'parametric equations'** of a surface. It is observed that parametric equations of a surface are not unique, that is, the surface (1) can be represented by different forms of the functions F_1, F_2, F_3 of the set (2).

As an example, the set of parametric equations

$$x = a \sin u \cos v \quad , \quad y = a \sin u \sin v \ , \quad z = a \cos u$$

and the set

$$x = \frac{a\left(1 - v^2\right)}{1 + v^2} \cos u \quad , \quad y = \frac{a\left(1 - v^2\right)}{1 + v^2} \sin u \ , \quad z = \frac{2av}{1 + v^2}$$

represent the spherical surface

$$x^2 + y^2 + z^2 = a^2.$$

A surface in three dimensional space can be considered as being generated by a curve. Indeed, a point whose coordinates verify equation (1) and which lies in the plane $z = k$ (k is parameter) has the coordinates satisfying the equations

$$z = k \quad , \quad f(x, y, k) = 0 \tag{3}$$

which shows that the point (x, y, z) lies on a curve Γ_k in the plane $z = k$.

Another example, if S is the sphere with $x^2 + y^2 + z^2 = a^2$, then points of S with $z = k$ have

$$z = k \quad , \quad x^2 + y^2 = a^2 - k^2,$$

which shows that Γ_k is a circle of radius $\left(a^2 - k^2\right)^{1/2}$. As k changes from $-a$ to a, each point of the sphere is covered by one such circle.

Curves in Three Dimensions

The curve given by the pair of equations (3) can be considered as the intersection of the surface (1) with the plane $z = k$. This idea can be generalized.

Let the surfaces S_1 and S_2 be given by the relations

$$F(x, y, z) = 0 \quad , \quad G(x, y, z) = 0,$$

respectively. If these surfaces have common points, the coordinates of these points satisfy a pair of equations

$$F(x, y, z) = 0 \quad , \quad G(x, y, z) = 0. \tag{4}$$

The surfaces S_1 and S_2 intersect in a curve C so that the locus of a point whose coordinates satisfy a pair of equations (4) is a curve in a space.

A curve may be represented by parametric equations as a surface. Any three equations of the form

$$x = f_1(t) \quad , \quad y = f_2(t) \quad , \quad z = f_3(t) \tag{5}$$

in which t is continuous variable, may be considered as the parametric equations of a curve.

Tangent of a Curve

We assume that P is any point on the curve

$$x = x(s) \quad , \; y = y(s) \quad , \; z = z(s) \tag{6}$$

which is characterized by the value s of the arc length. Then s is the distance P_0P of P from some fixed point P_0 measured along the curve. Similarly, if Q is a point at a distance δ_s along the curve from P, the distance P_0Q becomes $s + \delta_s$ and the coordinates of Q will be $\{x(s+\delta_s),\; y(s+\delta_s),\; z(s+\delta_s)\}$.

The distance δ_s is the distance from P to Q measured along the curve and is greater than δ_c, the length of the chord PQ. As Q approaches the point P, the difference $\delta_s - \delta_c$ becomes relatively less. Therefore, we shall confine

$$\lim_{\delta_s \to 0} \frac{\delta_c}{\delta_s} = 1. \tag{7}$$

On the other hand, the direction cosines of the chord PQ are

$$\left\{ \frac{x(s+\delta_s) - x(s)}{\delta_c}, \frac{y(s+\delta_s) - y(s)}{\delta_c}, \frac{z(s+\delta_s) - z(s)}{\delta_c} \right\}.$$

Dividing by increment δ_s and taking limit $\delta_s \to 0$ by use of the limit (7), the direction cosines of the tangent to the curve (6) at the point P are

$$\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right) \tag{8}$$

As δ_s tends to zero, the point Q tends to point P, and the chord PQ takes up the direction to the tangent to the curve at P.

Normal of a Surface

Assume that the curve C given by the equations (6) lies on the surface S whose equation is $F(x, y, z) = 0$ (Figure 5).

If

$$F(x(s), y(s), z(s)) = 0, \tag{9}$$

the point $(x(s), y(s), z(s))$ of the curve lies on this surface. Let the curve entirely on the surface, then (9) becomes an identity for all values of s.

If we differentiate the equation (9) with respect to s, we have

$$\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0, \tag{10}$$

which shows that the tangent T to the curve C at the point P is perpendicular to the vector

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right). \tag{11}$$

Also, this vector is perpendicular to the tangent to every curve lying on S and passing through P. This vector is called as **'Normal'** to the surface S at the point P.

If the equation of the surface S is given by

$$z = f(x, y)$$

and we denote

$$\frac{\partial z}{\partial x} = p, \; \frac{\partial z}{\partial y} = q, \tag{12}$$

then since $F = f(x, y) - z$, we have $F_x = p, \; F_y = q, \; F_z = -1$. Thus, unit normal to the surface at the point (x, y, z) is

$$\frac{(p, q, -1)}{\sqrt{p^2 + q^2 + 1}}. \tag{13}$$

Tangent of a Curve which is Intersection of Two Surfaces

The equation of the tangent plane Π_1 at the point $P(x, y, z)$ to the surface S_1 whose equation is $F(x, y, z) = 0$ is

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0 \tag{14}$$

where (X, Y, Z) are the coordinates of any other point of the tangent plane. Similarly, the equation of the tangent plane Π_2 at P to the surface S_2 whose equation is $G(x, y, z) = 0$ is

$$(X - x)\frac{\partial G}{\partial x} + (Y - y)\frac{\partial G}{\partial y} + (Z - z)\frac{\partial G}{\partial z} = 0. \tag{15}$$

The intersection L of the planes Π_1 and Π_2 is the tangent at P to the curve C, which is the intersection of S_1 and S_2.

From (14) and (15), the equations of the line L are

$$\frac{X - x}{F_y G_z - F_z G_y} = \frac{Y - y}{F_z G_x - F_x G_z} = \frac{Z - z}{F_x G_y - F_y G_x}. \tag{16}$$

Also, the direction ratios of the line L are

$$\{F_y G_z - F_z G_y, \; F_z G_x - F_x G_z, \; F_x G_y - F_y G_x\}$$

or

$$\left\{\frac{\partial(F, G)}{\partial(y, z)}, \frac{\partial(F, G)}{\partial(z, x)}, \frac{\partial(F, G)}{\partial(x, y)}\right\}. \tag{17}$$

Example 1 *The direction cosines of the tangent at the point* (x, y, z) *to the conic* $x^2 - y^2 + 2z^2 = 1,\ x + y + z = 1$ *are proportional to* $(-y - 2z, 2z - x, x + y)$.

$$F = x^2 - y^2 + 2z^2 - 1$$
$$G = x + y + z - 1$$

So, $\frac{\partial(F,G)}{\partial(y,z)} = \begin{vmatrix} -2y & 4z \\ 1 & 1 \end{vmatrix} = 2\,(-y - 2z)$, *etc. from (17).*