1.2. Simultaneous differential equations of the first order and first degree

Consider the systems of simultaneous differential equations of the first order and first degree of the type

$$\frac{dx_i}{dt} = f_i\left(t, x_1, x_2, ..., x_n\right) \quad ; \quad i = 1, 2, ..., n. \tag{1}$$

Here, the problem is to determine the functions $x_i = x_i(t)$ satisfying the initial condfitions $x_i(t_0) = a_i \quad (\ i = 1, 2, ..., n)\,.$

For example, a differential equation of the n-th order

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, ..., \frac{d^{n-1}x}{dt^{n-1}}\right) \tag{2}$$

can be written in the form

$$\begin{aligned} \frac{dx}{dt} &= y_1 \\ \frac{dy_1}{dt} &= y_2 \\ &\vdots \\ \frac{dy_{n-2}}{dt} &= y_{n-1} \\ \frac{dy_{n-1}}{dt} &= f\left(t, x, y_1, y_2, ..., y_{n-1}\right) \end{aligned}$$

which is special case of (1).

The system of (1) may be written in the form

$$\frac{dx_1}{f_1\left(t, x_1, x_2, ..., x_n\right)} = \frac{dx_2}{f_2\left(t, x_1, x_2, ..., x_n\right)} = ... = \frac{dx_n}{f_n\left(t, x_1, x_2, ..., x_n\right)} = dt,$$

which has important role in the theory of partial differential equations.

1.2.1. Simultaneous differential equations of the first order and first degree in three variables

Let P, Q, and R be functions of x, y, and z in the region $\Omega \subset \mathbb{R}^3$. Consider the systems in three variables

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{3}$$

The solutions of the equations (3) trace out curves such that at the point (x, y, z) the direction cosines of the curves are proportional to $(P, Q, R)\,.$

The existence and uniqueness of solutions of the equations of the type (3) is proved in the book [Shepley L. Ross, Differential Equations, John Wiley, 1974]

Theorem 1 *If the functions* $f_1(x,y,z)$ *and* $f_2(x,y,z)$ *are continuous in the region defined by* $|x-a|<p,\ \ |y-b|<r\ ,\ \ \ |z-c|<s$*, and if the functions satisfy a Lipschitz condition in the form*

$$|f_1(x,y,z)-f_1(x,\eta,\xi)| \leq A_1|y-\eta| + B_1|z-\xi|\,,$$
$$|f_2(x,y,z)-f_2(x,\eta,\xi)| \leq A_2|y-\eta| + B_2|z-\xi|$$

in the region, then in a suitable interval $|x-a|<h$ *there exists a unique pair functions* $y(x)$ *and* $z(x)$ *which are continuous and have continuous derivatives in that interval so that they satisfy the differential equation*

$$\frac{dy}{dx}=f_1(x,y,z)\quad,\quad \frac{dz}{dx}=f_2(x,y,z)\,,$$

which have $y(a)=b\ ,\ z(a)=c$*. Here* $a,b,$ *and* c *are arbitrary.*

According to the theorem, there exists a cylinder $y=y(x)$ passing through the point $(a,b,0)$ and a cylinder $z=z(x)$ passing through the point $(a,0,c)$ such that

$$\frac{dy}{dx}=f_1\quad,\quad \frac{dz}{dx}=f_2.$$

The solution of the pair of these equations consists of the set of common points of the cylinders $y=y(x)$ and $z=z(x)$, that is it consists of the curve of intersection Γ. This curve depends on choice of initial conditions, i.e., it is the curve both satisfying the pair of differential equations and passing through the point (a,b,c).

Since the numbers a,b,c are arbitrary, the general solution of the given pair equations will consists of the curves which are formed by the intersection of one-parameter system of cylinders containing $y=y(x)$ with another one-parameter system of cylinders of which $z=z(x)$ is a particular member. That is, the general solution of (3) is a two-parameter family of curves.

1.2.2. Methods of solution $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Consider

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{4}$$

From (4), if we can find two relations of the form

$$u_1(x,y,z)=c_1\quad;\quad u_2(x,y,z)=c_2 \tag{5}$$

which involve two arbitrary constants c_1 and c_2, then we can write a two-parameter family of curves satisfying the differential equations (4).

Method I.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=dk$$

$$\Rightarrow \frac{\lambda dx+\mu dy+\nu dz}{\lambda P+\mu Q+\nu R}=\frac{\lambda Pdk+\mu Qdk+\nu Rdk}{\lambda P+\mu Q+\nu R}=dk$$

$(\lambda, \mu, \nu$ arbitrary$)$

Sometimes, it is possible to choose λ, μ, ν such that $\lambda P+\mu Q+\nu R\equiv 0$. For such multipliers, it should be

$$\lambda dx+\mu dy+\nu dz=0.$$

If the expression $\lambda dx+\mu dy+\nu dz$ is an exact differential, then

$$\lambda dx+\mu dy+\nu dz=du$$
$$\Rightarrow u=c_1$$

holds.

Example 1. Find the integral curves of the equations

$$\frac{dx}{y\left(x+y\right)-az}=\frac{dy}{x\left(x+y\right)+az}=\frac{dz}{z\left(x+y\right)}.$$

Solution: If we choose $\lambda, \mu,$ and ν as $\lambda=\frac{1}{z}, \mu=\frac{1}{z}$ and $\nu=-\frac{x+y}{z^2}$, we obtain

$$\frac{\lambda dx+\mu dy+\nu dz}{\lambda P+\mu Q+\nu R}=\frac{\frac{1}{z}dx+\frac{1}{z}dy-\frac{x+y}{z^2}dz}{\frac{x}{z}\left(x+y\right)+a+\frac{y}{z}\left(x+y\right)-a-\frac{(x+y)^2}{z}}$$

$$=\frac{\frac{1}{z}dx+\frac{1}{z}dy-\frac{x+y}{z^2}dz}{0}$$

$$\Rightarrow \frac{1}{z}dx+\frac{1}{z}dy-\frac{x+y}{z^2}dz=0$$
$$\Rightarrow \frac{dx+dy}{x+y}-\frac{1}{z}dz=0$$
$$\Rightarrow \frac{d\left(x+y\right)}{x+y}-\frac{1}{z}dz=0$$
$$\Rightarrow \ln\left(x+y\right)-\ln z=\ln c_1$$
$$\Rightarrow \frac{x+y}{z}=c_1=u_1\left(x,y,z\right)$$

On the other hand, if we choose $\lambda=-x, \mu=y$, and $\nu=-a$, we obtain

$$\frac{ydy-xdx-adz}{xy\left(x+y\right)+ayz-xy\left(x+y\right)+axz-az\left(x+y\right)}=\frac{ydy-xdx-adz}{0}$$

$$\Rightarrow ydy - xdx - adz = 0$$
$$\Rightarrow \frac{y^2}{2} - \frac{x^2}{2} - az = \frac{c_2}{2}$$
$$\Rightarrow y^2 - x^2 - 2az = c_2 = u_2(x, y, z).$$

Hence, the integral curves of the given differential equations are the members of the two-parameter family

$$\frac{x+y}{z} = c_1 = u_1(x, y, z)$$

and

$$y^2 - x^2 - 2az = c_2 = u_2(x, y, z).$$

Method II.

For the multiplier λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2,

$$\frac{\lambda_1 dx + \mu_1 dy + \nu_1 dz}{\lambda_1 P + \mu_1 Q + \nu_1 R} = \frac{\lambda_2 dx + \mu_2 dy + \nu_2 dz}{\lambda_2 P + \mu_2 Q + \nu_2 R}.$$

If the expressions on the both sides are exact differential and say W_1 and W_2, then

$$dW_1 = dW_2 \Rightarrow W_1 = W_2 + C$$

is satisfied.

Example 2. Solve the equations

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}.$$

Solution: Each of these ratios is equal to

$$\frac{\lambda dx + \mu dy + \nu dz}{\lambda(y+z) + \mu(z+x) + \nu(x+y)}.$$

For suitable λ, μ, and ν constant multiplier, we can write

$$\frac{dx + dy + dz}{2(x+y+z)} = \frac{dx - dy}{y - x} = \frac{dx - dz}{z - x}.$$

From

$$\Rightarrow \frac{dx + dy + dz}{2(x+y+z)} = \frac{dx - dy}{y - x}$$
$$\Rightarrow \ln(x+y+z) + 2\ln(x-y) = \ln c_1$$
$$\Rightarrow u_1(x, y, z) = (x+y+z)(x-y)^2 = c_1.$$

From

$$\frac{dx + dy + dz}{2(x+y+z)} = \frac{dx - dz}{z - x},$$

it follows

$$u_2(x,y,z) = (x+y+z)(x-z)^2 = c_2.$$

Method III.

By using $u_1 = c_1$ which is obtained via one of the above methods, we can find $u_2 = c_2$.

Example 3. Find the integral curves of the equations

$$\frac{dx}{x} = \frac{dy}{y+z} = \frac{dz}{z+x^2}. \tag{6}$$

Solution: From

$$\begin{aligned}
&\frac{dx}{x} = \frac{dz}{z+x^2} \Rightarrow \frac{dz}{dx} - \frac{z}{x} = x \text{ (Linear Equ.)} \\
&\lambda = \frac{1}{x} \text{ (integrating factor)} \\
&\Rightarrow \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = 1 \\
&\Rightarrow \frac{d}{dx}\left(\frac{z}{x}\right) = 1 \\
&\Rightarrow \frac{z}{x} = x + c_1 \\
&\Rightarrow z = c_1 x + x^2 \qquad (7)
\end{aligned}$$

From (6), we have

$$\begin{aligned}
&\Rightarrow \frac{dy}{y+z} = \frac{dx}{x} \\
&\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{z}{x} \\
&\underset{From\ (7)}{\Rightarrow} \frac{dy}{dx} = \frac{y}{x} + x + c_1 \\
&\Rightarrow \frac{dy}{dx} - \frac{y}{x} = x + c_1 \text{(Linear Equ.)} \\
&\lambda = \frac{1}{x} \text{ (integrating factor)} \\
&\Rightarrow \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1 + \frac{c_1}{x} \\
&\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = 1 + \frac{c_1}{x} \\
&\Rightarrow \frac{y}{x} = c_1 \ln x + x + c_2 \\
&\Rightarrow y = c_1 x \ln x + x^2 + c_2 x \qquad (8)
\end{aligned}$$

The integral curves of the given differential equations (6) are determined by the equations (7) and (8).