1.3. Pfaffian Differential Equations

Let $F_i \quad (i = 1, 2, ..., n)$ be functions of independent variables $x_1, x_2, ..., x_n$. The expression

$$\sum_{i=1}^{n} F_i \left(x_1, x_2, ..., x_n\right) dx_i \tag{1}$$

is called a Pfaffian differential form. The equation

$$\sum_{i=1}^{n} F_i \left(x_1, x_2, ..., x_n\right) dx_i = 0 \tag{2}$$

is called Pfaffian differential equation.

1.3.1. Pfaffian Differential Equation In Two Variables

The Pfaffian differential equation in two variables is in the form

$$P\left(x, y\right) dx + Q\left(x, y\right) dy = 0 \tag{3}$$

which is equivalent to

$$\frac{dy}{dx} = f\left(x, y\right) \tag{4}$$

where $f\left(x, y\right) = -P/Q$.

$f\left(x, y\right)$ is defined uniquely at each point of $x0y$ plane at which $P\left(x, y\right)$ and $Q\left(x, y\right)$ are defined. If P and Q are single-valued, $\frac{dy}{dx}$ is single-valued. The solution of (3) satisfying $y\left(x_0\right) = y_0$ gives the curve which passes through $\left(x_0, y_0\right)$ and whose tangent at each point is defined by (4).

If we generalize this simple geometrical argument, the equation (3) defines a one-parameter family of curves in $x0y$ plane. That is, there exists a function $\phi\left(x, y\right)$ in a certain region of the $x0y$ plane such that

$$\phi\left(x, y\right) = c \tag{5}$$

defines a function $y\left(x\right)$ which satisfies identically the equation (3). Here, c is a constant.

If the differential form $Pdx + Qdy$ may be written in the form $d\phi\left(x, y\right)$, the equation (3) is called exact or integrable. If the form is not exact, it follows from (5)

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

that there exists a function $\lambda\left(x, y\right)$ such that

$$\frac{1}{P} \frac{\partial \phi}{\partial x} = \frac{1}{Q} \frac{\partial \phi}{\partial y} = \lambda.$$

If we multiply equation (3) by $\lambda(x, y)$, we can write

$$0 = \lambda(Pdx + Qdy) = d\phi$$

where $\lambda(x, y)$ is called an integrating factor of the pfaffian differential equation (3).

Theorem 1 *A pfaffian differential equation in two variables always has an integrating factor.*

1.3.2. Pfaffian Differential Equation In Three Variables

The pfaffian differential equation in three variables is in the form

$$Pdx + Qdy + Rdz = 0 \tag{6}$$

where P, Q, and R are functions of x, y, and z. By means of the vectors $X = (P, Q, R)$ and $dr = (dx, dy, dz)$, we can write the equation (6) in the vector notation as follows

$$X \cdot dr = 0 \tag{7}$$

Also,

$$\operatorname{curl} X = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Before considering this equation, we start with the following theorem:

Theorem 2 *If X is a vector such that $X \cdot \operatorname{curl} X = 0$ and λ is an arbitrary function of x, y, z then*

$$(\lambda X) \cdot \operatorname{curl}(\lambda X) = 0.$$

Now, we return to the pfaffian differential equation (6). It is not true that all equations in this form possess integral. But, if there exists a function $\lambda(x, y, z)$ such that

$$\lambda(Pdx + Qdy + Rdz)$$

is an exact differential $d\phi$ of a function $\phi(x, y, z)$, the equation (6) is called integrable. $\lambda(x, y, z)$ is called an integrating factor of the equation (6) and the function ϕ is called the primitive function of the differential equation.

Now, we give the next theorem to determine whether or not the equation (6) is integrable:

Theorem 3 *A necessary and sufficient condition that the pfaffian differential equation*

$$X \cdot dr = 0$$

should be integrable is that

$$X \cdot \operatorname{curl} X = 0.$$

Note: If λ is an integrating factor giving a solution $\phi = c$ and Φ is an arbitrary function of ϕ, then $\lambda\left(\frac{d\Phi}{d\phi}\right)$ is also integrating factor of the given equation. Since Φ is arbitrary, there are infinitely many integrating factors of this type.

We now consider methods for the solutions of pfaffian differential equations in three variables.

(a) By Inspection:

Example 1. Solve the equation

$$3yx^2dx + \left(y^2z - x^3\right)dy + y^3dz = 0,$$

firstly show that it is integrable.

Solution:

$$\begin{aligned} X &= \left(3yx^2, y^2z - x^3, y^3\right) \\ \operatorname{curl} X &= \left(2y^2, 0, -6x^2\right) \\ &\Rightarrow X \cdot \operatorname{curl} X = 0 \end{aligned}$$

So, the equation is integrable. We can write the equation in the form

$$\begin{aligned} &\Rightarrow y^2\left(zdy + ydz\right) - x^3dy + 3yx^2dx = 0 \\ &\Rightarrow zdy + ydz - \frac{x^3}{y^2}dy + \frac{3x^2}{y}dx = 0 \\ &\Rightarrow d\left(yz\right) + d\left(\frac{x^3}{y}\right) = 0 \end{aligned}$$

So the primitive of the equation is

$$\begin{aligned} yz + \frac{x^3}{y} &= c \\ &\Rightarrow y^2z + x^3 = cy \quad (c \text{ is a constant}) \end{aligned}$$

(b) Variables Separable:

In this case, such an equation is in the form

$$P\left(x\right)dx + Q\left(y\right)dy + R\left(z\right)dz = 0$$

$$\Rightarrow \int P\left(x\right)dx + \int Q\left(y\right)dy + \int R\left(z\right)dz = c$$

(c is a constant)

Example 2. Solve the equation

$$4y^2z^2dx + z^2x^2dy - x^2y^2dz = 0.$$

Solution: Dividing by $x^2y^2z^2$,

$$\frac{4}{x^2}dx + \frac{1}{y^2}dy - \frac{1}{z^2}dz = 0.$$

$$\int \frac{4}{x^2}dx + \int \frac{1}{y^2}dy - \int \frac{1}{z^2}dz = -c$$
$$\Rightarrow -\frac{4}{x} - \frac{1}{y} + \frac{1}{z} = -c$$

Integral surfaces are

$$\frac{4}{x} + \frac{1}{y} - \frac{1}{z} = c, \quad (c \text{ is a constant})$$

(c) One Variable Separable:
In this case, the equation is of the form

$$P(x,y)\,dx + Q(x,y)\,dy + R(z)\,dz = 0, \quad (\text{we say, } z \text{ is separable}).$$

$$X = (P,Q,R) \Rightarrow \operatorname{curl} X = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

The condition for integrability, $X \cdot \operatorname{curl} X = 0$, implies that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

So, $Pdx + Qdy$ is an exact differential, i.e. $Pdx + Qdy = du$.
Thus,

$$du + R(z)\,dz = 0,$$

the primitive of the equation is

$$u(x,y) + \int R(z)\,dz = c.$$

Example 3. Verify that the equation

$$y\left(x^2 - a^2\right)dy + x\left(y^2 - z^2\right)dx - z\left(x^2 - a^2\right)dz = 0$$

is integrable and solve it.

Solution: Dividing by $\left(x^2 - a^2\right)\left(y^2 - z^2\right)$,

$$\frac{ydy - zdz}{y^2 - z^2} + \frac{x}{x^2 - a^2}dx = 0.$$

It is separable in x. Since $\dfrac{\partial Q}{\partial z}=\dfrac{\partial R}{\partial y}$, it is integrable.

$$\Rightarrow \frac{1}{2}\frac{d\left(y^2-z^2\right)}{y^2-z^2}+\frac{1}{2}\frac{d\left(x^2-a^2\right)}{x^2-a^2}=0$$
$$\Rightarrow \frac{1}{2}\ln\left(y^2-z^2\right)+\frac{1}{2}\ln\left(x^2-a^2\right)=\frac{\ln c}{2}$$
$$\Rightarrow \left(y^2-z^2\right)\left(x^2-a^2\right)=c$$

(d) Homogeneous Equations:

The equation

$$P\left(x,y,z\right)dx+Q\left(x,y,z\right)dy+R\left(x,y,z\right)dz=0$$

is said to be homogeneous if the functions P,Q,R are homogeneous in x,y,z of the same degree n. To find the solution, we make the substitutions

$$y=ux \quad , \quad z=vx,$$

then we apply method (c).

Example 4. Verify that the equation

$$yz(y+z)dx+xz\left(x+z\right)dy+xy\left(x+y\right)dz=0$$

is integrable and find its solution.

Solution: The condition of integrability is satisfied. If we first make substitutions

$$y=ux,\ z=vx,$$
$$dy=udx+xdu \quad , \quad dz=vdx+xdv,$$

then we obtain

$$\frac{dx}{x}+\frac{v\left(v+1\right)du+u\left(u+1\right)dv}{2uv\left(u+v+1\right)}=0.$$

Splitting the factors of du and dv into partial fractions,

$$\frac{2dx}{x}+\left(\frac{1}{u}-\frac{1}{1+u+v}\right)du+\left(\frac{1}{v}-\frac{1}{1+u+v}\right)dv=0$$

$$\frac{2dx}{x}+\frac{du}{u}+\frac{dv}{v}-\frac{du+dv}{1+u+v}=0$$
$$\frac{2dx}{x}+\frac{du}{u}+\frac{dv}{v}-\frac{d\left(1+u+v\right)}{1+u+v}=0$$

$$\Rightarrow 2\ln x+\ln u+\ln v-\ln\left(1+u+v\right)=\ln c$$
$$\Rightarrow x^2uv=c\left(1+u+v\right) \quad , \quad c \text{ is a constant}$$

From $u=\dfrac{y}{x},\ v=\dfrac{z}{x}$, the solution is

$$xyz=c\left(x+y+z\right).$$