SECTION 3. HIGHER ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

In the previous sections, we analyzed the first order partial differential equations by studied them in different forms. In the following sections, we will examine two or higher order partial differential equations. Since the equations encountered in physics and engineering are generally second order linear partial differential equations, we will examine these types of equations and especially those with constant coefficients in this section.

Definition 1. If an operator L satisfies

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$$

for any constants c_1, c_2 and functions u_1, u_2, then L is called a *linear operator*. This definition can be extended to a finite number of functions as follows.
If u_1,...,u_n are functions and c_1,...,c_n are any real constants, then

$$L\left(\sum_{i=1}^{n} c_iu_i\right) = \sum_{i=1}^{n} c_iL(u_i).$$

The function $\sum_{i=1}^{n} c_iu_i$ is called a linear combination of u_1,...,u_n.

Definition 2. L is a linear partial derivative operator and f is any given function, then

$$L\left(u\right) = f$$

is called a linear partial differential equation. In this equation, if

$$f \equiv 0,$$

the equation is called a homogeneous linear partial differential equation, and if

$$f \neq 0,$$

the equation is called a non-homogeneous linear partial differential equation.
If each of the n functions u_1,..., u_n fulfill the homogen equation

$$L\left(u_i\right) = 0 \quad ; \qquad i = 1, ..., n$$

then any linear combination of these functions also satisfies the same equation. In other words

$$L\left(\sum_{i=1}^{n} c_iu_i\right) = 0$$

is provided. This important case for homogeneous linear partial differential equations is known as the superposition principle.

The opposite of this principle, which is frequently encountered in ordinary differential equations and partial differential equations, is as follows:

If the functions of v_1,...,v_n satisfy the equation as follows

$$L\left(v_i\right) = f_i \quad ; \qquad i = 1, ..., n$$

then the function $v = v_1 + ... + v_n$

$$L(v) = f_1 + ... + f_n$$

satisfy the equation.

As a result, we can give the following rule that applies to all linear partial differential equations: "General solution of a non-homogeneous linear partial differential equation is equal to the sum of the general solution of the homogeneous equation and any particular solution of the non-homogeneous equation."

As an example, consider the equation

$$u_{xx} - u_{yy} = 5\cos\left(2x + y\right) - 3\sin\left(2x + y\right).$$

The homogeneous part of this equation is

$$u_{xx} - u_{yy} = 0.$$

The corresponding solution to the homogeneous part is as follows

$$u_h(x, y) = f(x + y) + g(x - y)$$

where f and g are arbitrary twice differentiable functions.

A particular solution of the given equation is

$$u_p(x, y) = -\frac{5}{3}\cos\left(2x + y\right) + \sin\left(2x + y\right).$$

Thus, the general solution of the given equation is written

$$u = u_h + u_p$$

$$u(x, y) = f(x + y) + g(x - y) - \frac{5}{3}\cos\left(2x + y\right) + \sin\left(2x + y\right)$$

Here f and g are arbitrary twice differentiable functions.

As an another example, we consider

$$u_{xx} - u_{yy} = 3e^{x+y}.$$

The general solution of the homogeneous equation

$$u_{xx} - u_{yy} = 0$$

is in the form

$$u_h(x, y) = f(x + y) + g(x - y)$$

A particular solution of the given equation is

$$u_p = \frac{3}{2} x e^{x+y}.$$

Thus, the general solution of the given nonhomogeneous partial differential equation is as follows

$$u = f(x + y) + g(x - y) + \frac{3}{2} x e^{x+y}$$

where f and g are arbitrary twice differentiable functions.