1. Partial Differential Equations with variable coefficient

The general form of two-variable second-order linear partial differential equations are expressed as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y) \tag{1}$$

where x and y are independent variables and u is dependent variable. In (1), the coefficients A, B, C, D, E, F are functions of x and y and all A, B, C should not be zero at the same time. It will be assumed that the function u and the coefficients have continuous partial derivatives of second order in region R of the xy-plane. The function $G(x, y)$ must be a real valued function in the region R. It is not always possible to obtain the general solution of equation (1), even if it has a constant coefficient. In this section, the situations, in which general solutions of equation (1) can be obtained, will be examined. We use the notation

$$u_{xx} = r \quad , \qquad u_{xy} = s \quad , \qquad u_{yy} = t \quad , \qquad u_x = p \quad , \qquad u_y = q$$

1.1. Special Types

In the special cases where some of the coefficients A, B, C, D, E and F are zero in equation (1), the general solution of the equation can be easily obtained by simple integration processes or by converting them into ordinary differential equations. Let's see them now.

I. If $A \neq 0,\ B = C = D = E = \ F = 0$, to obtain the general solution of the special equation

$$A(x, y)u_{xx} = G(x, y) \tag{2}$$

it is sufficient to integrate both sides of (2) twice with respect to x. During this process, the variable y will be considered as a parameter. Similarly, for the equations

$$B(x, y)u_{xy} = G(x, y) \tag{3}$$

and

$$C(x, y)u_{yy} = G(x, y,) \tag{4}$$

similar methods are used to obtain general solutions of the equations. It should also be kept in mind that equations (2), (3) and (4) can be solved by converting them into linear equations with constant coefficients.

Example 1. Find the general solution of the equation $r = 6xy^2 - 2y$.

Solution: The given partial differential equation is written as follows

$$\frac{\partial^2 u}{\partial x^2} = 6xy^2 - 2y$$

and if both sides are integrated twice with respect to x, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2y^2 - 2xy + f(y) \\ u(x, y) &= x^3y^2 - x^2y + xf(y) + g(y) \end{aligned}$$

where f and g are arbitrary functions.

Example 2. Find the general solution of the equation $yu_{xy} - x + ay = 0$, (a constant).

Solution: If the given partial differential equation is written as

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x}{y} - a$$

and then we integrate once with respect to each of the x and y variables, we have the general solution

$$\begin{aligned} \frac{\partial u}{\partial x} &= x \ln y - ay + f_1(x) \\ u(x,y) &= \frac{1}{2}x^2 \ln y - axy + f(x) + g(y), \end{aligned}$$

where f_1, f and g are arbitrary functions and $\int f_1 \, dx = f$ is taken.

Example 3. Find the general solution of the equation $x^m y^n z_{yy} = ax^m + by^n$. Here a, b, m, n are real constants and $n \neq 1, 2$.

Solution: The given partial differential equation is written as follows

$$\frac{\partial^2 z}{\partial y^2} = \frac{a}{y^n} + \frac{b}{x^m}.$$

If we integrate it twice with respect to y, we have the general solution

$$\begin{aligned} \frac{\partial z}{\partial y} &= a\frac{y^{-n+1}}{-n+1} + \frac{b}{x^m}y + f(x) \\ z(x,y) &= \frac{a}{-n+1}\,\frac{y^{-n+2}}{-n+2} + \frac{b}{x^m}\,\frac{y^2}{2} + yf(x) + g(x) \end{aligned}$$

or

$$z(x,y) = \frac{a}{(1-n)(2-n)}\,\frac{1}{y^{n-2}} + \frac{b}{2}\,\frac{y^2}{x^m} + yf(x) + g(x)$$

where f and g are arbitrary functions.

II. If $A \neq 0$, $D \neq 0$, $B = C = E = F = 0$, we have the special case of the equation as follows

$$Au_{xx} + Du_x = G \qquad (5)$$

To obtain the general solution of the equation, it is sufficient to transform $u_x = v$. In this case we have $u_{xx} = v_x$ and the equation (5) reduces to the equation

$$Av_x + Dv = G.$$

This last equation can be solved as an ordinary linear differential equation with an independent variable x. The general solution of (5) $u(x,y)$ is found by integrating the obtained function $v = v(x,y)$ with respect to x. During these operations, the variable y will act as a parameter. Similarly, for the equations

$$\begin{aligned} Bu_{xy} + Du_x &= G \quad (6) \\ Bu_{xy} + Eu_y &= G \quad (7) \\ Cu_{yy} + Eu_y &= G \quad (8) \end{aligned}$$

similar methods are used to obtain general solutions of the equations.

Example 4. Find the general solution of the equation $xu_{xy} + u_y = 4e^{2x-y}$.

Solution: For $u_y = v$, we have $u_{xy} = v_x$ and the given partial differential equation reduces to the equation

$$xv_x + v = 4e^{2x-y}.$$

By writing this last equation as

$$x\frac{\partial v}{\partial x} + v = 4e^{2x}e^{-y}$$

it can be solved as an ordinary linear differential equation or we can write this equation as

$$\frac{\partial}{\partial x}(xv) = 4e^{-y}e^{2x}.$$

If we integrate both sides with respect to x, we have

$$xv = 4e^{-y}\frac{1}{2}e^{2x} + f_1(y)$$

and

$$v = \frac{2}{x}e^{2x-y} + \frac{1}{x}f_1(y).$$

Since $v = u_y$, we obtain the general solution of the equation as

$$\begin{aligned} u &= \int v\, dy = \int \left[\frac{2}{x}e^{2x-y} + \frac{1}{x}f_1(y)\right] dy \\ &= -\frac{2}{x}e^{2x-y} + \frac{1}{x}f(y) + g(x) \end{aligned}$$

were f_1, f and g are arbitrary functions and $\int f_1\, dx = f$.

III. In the condition of $A \neq 0,\ B \neq 0,\ D \neq 0,\ C = E = \ F = 0$, we have the special case of the equation as follows

$$Au_{xx} + Bu_{xy} + Du_x = G \quad (9)$$

To obtain the general solution of the equation, it is sufficient to say $u_x = v$. In this case we have $u_{xx} = v_x$ and $u_{xy} = v_y$, and the equation (9) reduces to the equation

$$Av_x + Bv_y + Dv = G. \quad (10)$$

After obtaining the solution $v = v(x, y)$ from equation (10), which is a first-order linear partial differential equation, by known methods, integrating v with respect to the variable x gives the general solution $u = u(x, y)$ of (9). Based on this idea, equation

$$Bu_{xy} + Cu_{yy} + Eu_y = G$$

is solved in the same way by saying $u_y = v$.

Example 5. Find the general solution of the equation $xr - ys + p = y^2$.

Solution: The partial differential equation given is written explicitly as

$$xu_{xx} - yu_{xy} + u_x = y^2$$

If we take $u_x = p$, then $u_{xx} = \dfrac{\partial p}{\partial x}$, $u_{xy} = \dfrac{\partial p}{\partial y}$ and the given partial differential equation reduces to equation

$$x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y} = y^2 - p.$$

The Lagrange system corresponding to last equation is as follows

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dp}{y^2 - p}.$$

The first integrals are

i) $\dfrac{dx}{x} = \dfrac{dy}{-y} \Rightarrow \ln x + \ln y = \ln c_1 \Rightarrow xy = c_1$

ii) $ydp - pdy + y^2 dy = 0 \Rightarrow \dfrac{ydp - pdy}{y^2} + dy = 0$

$\Rightarrow d\left(\dfrac{p}{y} + y\right) = 0 \Rightarrow \dfrac{p}{y} + y = c_2.$

From $c_2 = f_1(c_1)$, we have the general solution

$$\frac{p}{y} + y = f_1(xy)$$

or

$$p = -y^2 + yf_1(xy)$$

For $p = \dfrac{\partial u}{\partial x}$, we can write

$$u = \int p\,dx$$

and we find the general solution as

$$\begin{aligned} u(x, y) &= \int \left[-y^2 + y\,f_1(xy)\right]\,dx \\ &= -y^2x + f(xy) + g(y) \end{aligned}$$

where f and g are arbitrary functions.

IV. In special case $A \neq 0\,, \quad D \neq 0\,, \quad F \neq 0\,, \quad B = C = E = 0$, we have the equation

$$Au_{xx} + Du_x + Fu = G \tag{11}$$

For the solution of this equation, It is taken into account as an ordinary linear differential equation of second order where y plays the role of a parameter and x is the independent variable. In similar way, we can solve the equation

$$Cu_{yy} + Eu_y + Fu = G. \tag{12}$$

Example 6. Find the general solution of the equation $u_{xx} - 3yu_x + 2y^2u = 2(y-2)e^{2x-y}$.

Solution: In the given partial differential equation there are only derivatives with respect to the variable x . It can be solved as an second order ordinary linear differential equation. During these operations, the variable y is considered as a parameter. The characteristic equation for the homogeneous part is

$$\lambda^2 - 3y\lambda + 2y^2 = 0$$

and we can find

$$\lambda_{1,2} = \frac{3y \mp \sqrt{9y^2 - 8y^2}}{2} = \frac{3y \mp y}{2} \qquad \Rightarrow \qquad \lambda_1 = y \quad , \qquad \lambda_2 = 2y$$

In that case, we obtain the general solution of the homogeneous part

$$u_h = e^{yx}f(y) + e^{2yx}g(y).$$

Now let's find a particular solution u_p for the non-homogeneous equation. Let's choose u_p

$$u_p = A(y)e^{2x-y}$$

and determine $A(y)$ as follows

$$\begin{aligned} \frac{\partial}{\partial x}(u_p) &= 2A(y)e^{2x-y} \\ \frac{\partial^2}{\partial x^2}(u_p) &= 4A(y)e^{2x-y}. \end{aligned}$$

if they are put in place in the given equation, we have

$$\begin{aligned} 4A(y)e^{2x-y} - 3y\left(2A(y)e^{2x-y}\right) + 2y^2A(y)e^{2x-y} &= 2(y-2)e^{2x-y} \\ A(y)e^{2x-y}(4 - 6y + 2y^2) &= 2(y-2)e^{2x-y} \\ 2A(y)(2 - 3y + y^2) &= 2(y-2) \\ A(y)(y-1)(y-2) &= (y-2) \\ A(y) &= \frac{1}{y-1}. \end{aligned}$$

So, we can find

$$u_p = \frac{1}{y-1} e^{2x-y}$$

Thus, the general solution of the given partial differential equation is obtained as follows ($u(x,y) = u_h + u_p$)

$$u(x,y) = e^{xy} f(y) + e^{2xy} g(y) + \frac{e^{2x-y}}{y-1}$$

where f and g are arbitrary functions.

V. Euler-Poisson-Darboux (E.P.D.) Equation

Linear partial differential equation with variable coefficients

$$(x-y)u_{xy} - u_x + u_y = 0 \tag{13}$$

is called Euler-Poisson-Darboux equation. If we say

$$w = (x-y)u \Rightarrow u = x-y)^{-1} w,$$

we have

$$\begin{aligned} u_x &= (x-y)^{-1} w_x - (x-y)^{-2} w \\ u_y &= (x-y)^{-1} w_y + (x-y)^{-2} w \\ u_{xy} &= (x-y)^{-1} w_{xy} + (x-y)^{-2} w_x - (x-y)^{-2} w_y - 2(x-y)^{-3} w \\ &= (x-y)^{-1} w_{xy} + (x-y)^{-2} (w_x - w_y) - 2(x-y)^{-3} w. \end{aligned}$$

If we put these derivative in the equation (13), we find

$$\begin{aligned} (x-y)u_{xy} - u_x + u_y = w_{xy} + (x-y)^{-1}(w_x - w_y) - 2(x-y)^{-2} w \\ - (x-y)^{-1} w_x + (x-y)^{-2} w + (x-y)^{-1} w_y + (x-y)^{-2} w &= 0, \end{aligned}$$

or $w_{xy} = 0$. The general solution of $w_{xy} = 0$ is given by

$$w = f(x) + g(y)$$

and the general solution of E.P.D. is as follows

$$u(x,y) = \frac{1}{x-y} \left[f(x) + g(y) \right]$$

Here f and g are arbitrary functions.

Example 7. What are the values of the constants α and β in terms of constants a and b for the general solution of the equation $(ax - by)u_{xy} - \alpha u_x + \beta u_y = 0$ is available. (the special case $a = 1$, $b = 1$, $\alpha = 1$, $\beta = 1$ is E.P.D. equation.)

Solution: Let's say

$$w = (ax - by)u$$

or

$$u = \frac{w}{ax - by} = (ax - by)^{-1} w$$

in the given partial differential equation. If derivatives are taken, we have

$$\begin{aligned}
u_x &= (ax - by)^{-1} w_x - a(ax - by)^{-2} w \\
u_y &= (ax - by)^{-1} w_y + b(ax - by)^{-2} w \\
u_{xy} &= (ax - by)^{-1} w_{xy} + b(ax - by)^{-2} w_x - a(ax - by)^{-2} w_y - 2ab(ax - by)^{-3} w.
\end{aligned}$$

If these are put in place in the given equation, we can write

$$\begin{aligned}
(ax - by)u_{xy} - \alpha u_x + \beta u_y &= w_{xy} + b(ax - by)^{-1} w_x - a(ax - by)^{-1} w_y - 2ab(ax - by)^{-2} w \\
&\quad -\alpha(ax - by)^{-1} w_x + \alpha a(ax - by)^{-2} w + \beta(ax - by)^{-1} w_y \\
&\quad +\beta b(ax - by)^{-2} w \\
&= w_{xy} + (ax - by)^{-1} \left[(b - \alpha) w_x + (\beta - a) w_y\right] \\
&\quad +(ax - by)^{-2} \left[a\alpha - 2ab + \beta b\right] w \\
&= 0
\end{aligned}$$

From the last equality, we obtain $a\alpha - 2ab + \beta b = 0$, $b - \alpha = 0$ and $\beta - a = 0$. From the second and third equations, we can choose $\alpha = b$ and $\beta = a$ and these values satisfy the first equation. In that time, taking

$$\alpha = b \quad , \quad \beta = a$$

we find the equation

$$w_{xy} = 0$$

and the general solution can be written

$$w = f(x) + g(y).$$

The general solution of the equation is

$$u(x, y) = \frac{1}{ax - by} \left[f(x) + g(y)\right].$$

1.2. Euler Type Equation

Euler type equations with variable coefficient that can be converted into a linear equation with a constant coefficient with a suitable transformation are as follows

$$F(D_x, D_y)u = \left(\sum_{j,k} c_{jk}\, x^j y^k D_x^j D_y^k \right) u = G(x, y) \tag{14}$$

Here for $j, k = 0, 1, ..., m$, c_{jk} are constant. For example, an second order Euler type equation can be expressed as follows.

$$c_{20}x^2u_{xx} + c_{11}xyu_{xy} + c_{02}y^2u_{yy} + c_{10}xu_x + c_{01}yu_y + c_{00}u = G(x, y) \quad (15)$$

For the new independent variables ξ and η , If we use the transformation in (14),

$$x = e^{\xi} \quad , \quad y = e^{\eta}, \quad (16)$$

a linear partial differential equation with constant coefficients is obtained. The partial derivative operators are given by

$$\frac{\partial}{\partial x} = D_x \quad , \quad \frac{\partial}{\partial y} = D_y \quad , \quad \frac{\partial}{\partial \xi} = D_\xi \quad , \quad \frac{\partial}{\partial \eta} = D_\eta$$

Under the substitutions (16), it follows

$$\begin{aligned}
D_x u &= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{x}D_\xi u \quad \Rightarrow \quad xD_x u = D_\xi u \\
D_y u &= \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{y}D_\eta u \quad \Rightarrow \quad yD_y u = D_\eta u \\
D_x^2 u &= \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial u}{\partial \xi}\right) = \frac{1}{x^2}\frac{\partial^2 u}{\partial \xi^2} - \frac{1}{x^2}\frac{\partial u}{\partial \xi} = \frac{1}{x^2}\left(D_\xi^2 - D_\xi\right)u \quad \Rightarrow \quad x^2D_x^2 u = D_\xi\left(D_\xi - 1\right)u \\
D_xD_y u &= \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial u}{\partial \eta}\right) = \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right) = \frac{1}{xy}\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{xy}D_\xi D_\eta u \quad \Rightarrow \quad xyD_xD_y u = D_\xi D_\eta u \\
D_y^2 u &= \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial u}{\partial \eta}\right) = \frac{1}{y^2}\frac{\partial^2 u}{\partial \eta^2} - \frac{1}{y^2}\frac{\partial u}{\partial \eta} = \frac{1}{y^2}\left(D_\eta^2 - D_\eta\right)u \quad \Rightarrow \quad y^2D_y^2 u = D_\eta\left(D_\eta - 1\right)u
\end{aligned}$$

By induction, we obtain

$$x^jy^kD_x^jD_y^k u = D_\xi\left(D_\xi - 1\right)...\left(D_\xi - j + 1\right)D_\eta\left(D_\eta - 1\right)...\left(D_\eta - k + 1\right)u \quad (17)$$

If we write these derivatives in the Euler equation, we find a linear equation with constant coefficients. Firstly, we solve this linear equation and then from $\xi = \ln x$, $\eta = \ln y$, we obtain the desired solution of Euler equation.

Example 8. Find the general solution of the equation $x^2u_{xx} + 3xyu_{xy} + 2y^2u_{yy} + xu_x + 2yu_y = xy$.

Solution: The given partial differential equation is an Euler type equation. Let's apply the substitutions $x = e^{\xi}$, $y = e^{\eta}$, we have the following derivatives

$$\begin{array}{lll}
u_x = D_x u = \frac{1}{x}D_\xi u & \Rightarrow & xu_x = D_\xi u \\
u_y = D_y u = \frac{1}{y}D_\eta u & \Rightarrow & yu_y = D_\eta u \\
u_{xx} = \frac{1}{x^2}\left(D_\xi^2 - D_\xi\right)u & \Rightarrow & x^2u_{xx} = D_\xi\left(D_\xi - 1\right)u \\
u_{xy} = \frac{1}{xy}D_\xi D_\eta u & \Rightarrow & xyu_{xy} = D_\xi D_\eta u \\
u_{yy} = D_y^2 u = \frac{1}{y^2}\left(D_\eta^2 - D_\eta\right)u & \Rightarrow & y^2u_{yy} = D_\eta\left(D_\eta - 1\right)u.
\end{array}$$

If these are put in place in the given equation

$$F(D_\xi, D_\eta)u = [D_\xi (D_\xi - 1) + 3D_\xi D_\eta + 2D_\eta (D_\eta - 1) + D_\xi + 2D_\eta] u = e^\xi e^\eta$$

and doing some simplification, we have

$$F(D_\xi, D_\eta)u = \left(D_\xi^2 + 3D_\xi D_\eta + 2D_\eta^2\right) u = (D_\xi + D_\eta)(D_\xi + 2D_\eta)u = e^{\xi+\eta}.$$

The solution of the homogeneous part of this equation is as follows

$$u_h = f(\xi - \eta) + g(2\xi - \eta).$$

A particular solution u_p for the non-homogeneous equation is given as

$$u_p = \frac{1}{F(D_\xi, D_\eta)} e^{\xi+\eta} = \frac{e^{\xi+\eta}}{F(1,1)} = \frac{e^{\xi+\eta}}{(1+1)(1+2)} = \frac{1}{6} e^{\xi+\eta}$$

Thus we find the general solution of the equation with constant coefficient as

$$u = u_h + u_p$$

and

$$u = f(\xi - \eta) + g(2\xi - \eta) + \frac{1}{6} e^{\xi+\eta}$$

Here f and g are arbitrary functions. If $\xi = \ln x$ ve $\eta = \ln y$ are written in the general solution, we can obtain general solution of the given equation

$$\begin{aligned} u &= f(\ln x - \ln y) + g(2\ln x - \ln y) + \frac{1}{6} xy \\ &= f[\ln(\frac{x}{y})] + g[\ln(\frac{x^2}{y})] + \frac{1}{6} xy \\ &= F(\frac{x}{y}) + G(\frac{x^2}{y}) + \frac{1}{6} xy \end{aligned}$$

Here $F = f o \ln$, $G = f o \ln$ are arbitrary functions.