2. Classification of Partial Differential Equations

It is known from analytical geometry that the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{1}$$

in x and y indicates a curve that can be the intersection of the cone and the plane, i.e., a conic equation. According to whether $\Delta = b^2 - 4ac$ is positive, zero, or negative, equation (1) defines the equations for hyperbola, parabola, or ellipse (or the degenerate form of any of these) where $a,\ b,\ c,\ d,\ e$ and f are constants. With the help of a linear transformation of the x and y variables, equation (1) can be converted to its standard (central) form by using the new variables ξ and η. The $\xi\eta$-coordinate system is a relative coordinate system in which the equation of the curve takes the simplest and the properties of the given curve remain the same.
An approach similar to the above classification can also be used for second order linear partial differential equations. Now consider the second order linear partial differential equation with variable coefficient

$$Lu = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y) \tag{2}$$

with two independent variables. Let the function u and the coefficients $A,\ B,\ C$ be real valued functions having second-order continuous derivatives in a region R of the xy-plane. and $A,\ B,\ C$ all not be zero at the same time. The sum of the terms in equation (2)

$$Au_{xx} + Bu_{xy} + Cu_{yy} \tag{3}$$

is called the *fundamental part* of equation (2) and this part determines the properties of the solution of the equation u. In R, the function Δ is called the discriminant of equation (2) or the operator L.

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)\, C(x, y) \tag{4}$$

At a certain point $(x_0, y_0) \in R$,

$$\Delta(x_0, y_0) = B^2(x_0, y_0) - 4A(x_0, y_0)\, C(x_0, y_0).$$

According to whether the discriminant is positive, zero or negative, it is said that equation (2) or L operator is hyperbolic, parabolic or elliptical type at the point (x_0, y_0). That is, equation (2) at point $(x, y) \in R$

i) If $\Delta(x, y) > 0$ then hyperbolic type

ii) If $\Delta(x, y) = 0$ then parabolic type

iii) If $\Delta(x, y) < 0$ then elliptic type

If equation (2) is hyperbolic at every point in the region R, then the equation is called hyperbolic in R. In the same way cases of being parabolic in R or elliptical in R correspond to being parabolic or elliptical at every point of R. In general, the L operator or the equation $Lu = G$ can be of all three types in the domain of the coefficients. If the coefficients $A,\ B,\ C$ are constants, since the sign of Δ will be the same in the whole plane, the corresponding equation will be of only one type in the whole plane.

Example 1. Let's consider the equation $(1-x^2)u_{xx} - 2xyu_{xy} + (1-y^2)u_{yy} + xu_x + 3x^2yu_y - 2u = 0$. The discriminant of this equation is as follows

$$\Delta = (-2xy)^2 - 4(1-x^2)(1-y^2) = 4(-1+x^2+y^2).$$

The given partial differential equation is hyperbolic in $x^2+y^2 > 1$, parabolic on the circle $x^2+y^2 = 1$, and elliptical type inside the circle $x^2+y^2 < 1$. (Figure 2.1).

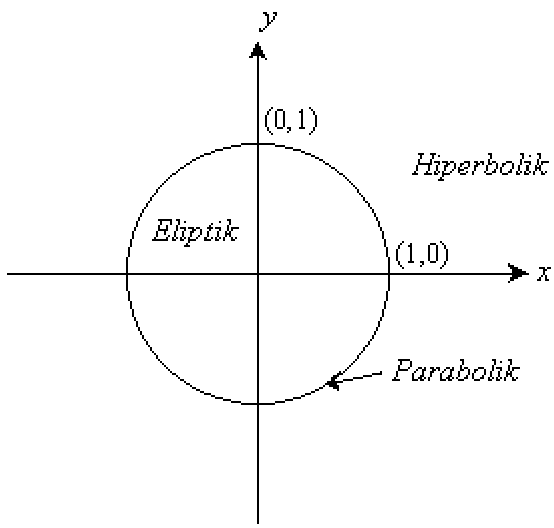

Figure 2.1

Example 2. Let's determine the type of equation $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 0$. The given partial differential equation has a constant coefficient and the discriminant of the equation is following

$$\Delta = B^2 - 4AC = 5^2 - 4.4.1 = 25 - 16 = 9 > 0.$$

The equation is of the hyperbolic type at every point of the xy-plane.

Theorem: The type of equation (2) does not change under continuous, one-to-one and real transformations of independent variables

$$\xi = \xi(x,y) \quad , \qquad \eta = \eta(x,y).$$

2.1. Canonical Forms

In this section we will see how to convert equation (2) into its canonical form, also called its normal form or standard form. Let's assume that none of the A, B and C are zero, initially.

The quadratic ordinary differential equation

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{5}$$

is called characteristic equation of the linear partial differential equation (2). The ordinary differential equations which are corresponding to the roots of equation (5) are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}, \tag{6}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}. \tag{7}$$

The families of curves that are solutions of equations (6) and (7) are called *characteristic curves* of equation (2). The solutions of (6) and (7), which are the ordinary differential equations of the first order, can be expressed as

$$\begin{array}{rclcl} \phi_1(x,y) & = & c_1 & , & c_1 = sabit \\ \phi_2(x,y) & = & c_2 & , & c_2 = sabit \end{array}$$

Thus, the change of variables

$$\xi = \phi_1(x,y) \quad , \quad \eta = \phi_2(x,y)$$

will convert the equation (2) into its canonical form.

A. Hyperbolic Type:

If $B^2 - 4AC > 0$, then two real different characteristic curve families from equations (6) and (7)

$$\begin{array}{rclcl} \phi_1(x,y) & = & c_1 & , & c_1 = sabit \\ \phi_2(x,y) & = & c_2 & , & c_2 = sabit \end{array}$$

are obtained. Under the substitutions

$$\xi = \phi_1(x,y) \quad , \quad \eta = \phi_2(x,y),$$

the partial differential equation (*2*) transforms into form

$$u_{\xi\eta} = H_1(\xi, \eta, u, u_\xi, u_\eta). \tag{8}$$

The equation (8) is called *the first canonical form of the hyperbolic equation.* Let α and β be new independent variables,

$$\alpha = \xi + \eta \quad , \quad \beta = \xi - \eta$$

If this change of variable is applied again to equation (8), we have

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta) \tag{9}$$

and this is called the *second canonical form of the hyperbolic equation.*

Example 1. Find the canonical form of the equation $y^2 u_{xx} - x^2 u_{yy} = 0$.

Solution: For the given partial differential equation, we can write

$$A = y^2 \ , \quad B = 0 \ , \quad C = -x^2$$

then

$$\Delta = B^2 - 4AC = 4x^2y^2 > 0$$

So the equation for $x \neq 0\,,\ \ y \neq 0$ is of the hyperbolic type and for $x = 0$ or $y = 0$ the equation is of the parabolic type. Substituting A, B, C in the characteristic equation which is given by

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

we have ordinary differential equation

$$y^2\left(\frac{dy}{dx}\right)^2 - x^2 = 0$$

or

$$\frac{dy}{dx} = \pm\frac{x}{y}$$

By integrating these two equations, curve families are obtained

$$\frac{y^2}{2} - \frac{x^2}{2} = c_1 \qquad \text{and} \qquad \frac{y^2}{2} + \frac{x^2}{2} = c_2$$

which are the characteristic curves of the given partial differential equation. To convert the given partial differential equation to its canonical form, the change of variables

$$\xi = \frac{y^2}{2} - \frac{x^2}{2}$$
$$\eta = \frac{y^2}{2} + \frac{x^2}{2}$$

should be done. Under these change of variables, the derivatives will be obtained in terms of ξ and η as follows

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = -xu_\xi + xu_\eta, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y = yu_\xi + yu_\eta, \end{aligned}$$

$$
\begin{aligned}
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} \\
&= x^2u_{\xi\xi} - 2x^2u_{\xi\eta} + x^2u_{\eta\eta} - u_\xi + u_\eta,
\end{aligned}
$$

$$
\begin{aligned}
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy} \\
&= y^2u_{\xi\xi} + 2y^2u_{\xi\eta} + y^2u_{\eta\eta} + u_\xi + u_\eta.
\end{aligned}
$$

If these derivative values are written in the given partial differential equation, we obtain the canonical form of the equation

$$
u_{\xi\eta} = \frac{\eta}{2(\xi^2 - \eta^2)}u_\xi - \frac{\xi}{2(\xi^2 - \eta^2)}u_\eta.
$$

B. Parabolic Type:

Since $B^2 - 4AC = 0$, from (6) and (7) we get a single integral curve family in the form ξ =constant (or η=constant). We can choose any function $\eta = \eta(x, y)$ independent of $\xi(x, y)$. It is sufficient to choose $\eta = y$ for simplicity. After applying the specified change of variable, the given equation turns to

$$
u_{\eta\eta} = H_3(\xi, \eta, u, u_\xi, u_\eta) \tag{10}
$$

This equation is called *the canonical form of the parabolic equation.* On the other hand, by choosing $\xi = \phi(x, y)$ arbitrarily, we obtain

$$
u_{\xi\xi} = H_4(\xi, \eta, u, u_\xi, u_\eta),
$$

which is another canonical form of the parabolic equation.

Example 2. Obtain the canonical form of the equation $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$ and find its general solution.

Solution: Since $\Delta = B^2 - 4AC = (2xy)^2 - 4x^2y^2 = 0$, the equation is of the parabolic type everywhere. We have the characteristic equation as follows

$$
\begin{aligned}
A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C &= x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + y^2 \\
&= \left(x\frac{dy}{dx} - y\right)^2 = 0
\end{aligned}
$$

and

$$
\frac{dy}{y} - \frac{dx}{x} = 0.
$$

So, we obtain following family of real characteristic curves

$$
\ln y - \ln x = \ln c_1 \quad \Rightarrow \quad \frac{y}{x} = c_1
$$

So we can take one of the characteristic coordinates, for example here $\xi = \dfrac{y}{x}$. since we can arbitrarily take η independent of ξ, let's choose $\eta = y$ for simplicity. Thus, the given equation transforms into its canonical form under the substitutions $\xi = \dfrac{y}{x}$, $\eta = y$. If we calculate the derivatives in terms of characteristic coordinates ξ and η, we get

$$\begin{aligned}
u_x &= u_\xi\left(-\frac{y}{x^2}\right) + u_\eta.0 = -\frac{y}{x^2}u_\xi \\
u_y &= u_\xi\left(\frac{1}{x}\right) + u_\eta.1 = \frac{1}{x}u_\xi + u_\eta \\
u_{xx} &= -\frac{y}{x^2}u_{\xi\xi}\left(-\frac{y}{x^2}\right) + \left(-\frac{y}{x^2}\right)u_{\xi\eta}.0 + \frac{2y}{x^3}u_\xi = \frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi \\
u_{xy} &= -\frac{y}{x^2}u_{\xi\xi}\left(\frac{1}{x}\right) + \left(-\frac{y}{x^2}\right)u_{\xi\eta}.1 - \frac{1}{x^2}u_\xi = -\frac{y}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\xi
\end{aligned}$$

$$u_{yy} = \frac{1}{x}u_{\xi\xi}\left(\frac{1}{x}\right) + \frac{1}{x}u_{\xi\eta} + u_{\eta\xi}\left(\frac{1}{x}\right) + u_{\eta\eta}.1 = \frac{1}{x^2}u_{\xi\xi} + \frac{2}{x}u_{\xi\eta} + u_{\eta\eta}.$$

If these results are written in the given partial differential equation, we obtain

$$\begin{aligned}
x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} &= x^2\left[\frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi\right] + 2xy\left[-\frac{y}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\xi\right] \\
&\quad + y^2\left[\frac{1}{x^2}u_{\xi\xi} + \frac{2}{x}u_{\xi\eta} + u_{\eta\eta}\right] \\
&= \underbrace{\left(\frac{y^2}{x^2} + \frac{y^2}{x^2} - \frac{2y^2}{x^2}\right)}_{=0}u_{\xi\xi} + \underbrace{\left(-\frac{2y^2}{x} + \frac{2y^2}{x}\right)}_{=0}u_{\xi\eta} + y^2u_{\eta\eta} + \underbrace{\left(\frac{2y}{x} - \frac{2y}{x}\right)}_{=0}u_\xi \\
&= 0
\end{aligned}$$

or shortly we have the following canonical form

$$u_{\eta\eta} = 0.$$

To find the general solution of this equation, it is sufficient to integrate with respect to η. Then, we have

$$u = f(\xi) + \eta g(\xi)$$

Substituting the values of ξ and η in terms of x and y, the general solution of the given equation $u = u(x, y)$ is obtained as follows

$$u = f(\frac{y}{x}) + y\,g(\frac{y}{x}),$$

where f and g are arbitrary functions.

C. Elliptic Type:

If equation (2) is of elliptic type, we know that $B^2 - 4AC < 0$. In that case,

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

the characteristic equation can not have real solutions but it has two complex conjugate solutions consisting of complex valued and continuous functions of x and y variables so there are no real characteristic curves for elliptic type partial differential equations. In this case, consider

$$\left.\begin{array}{l} \xi = \alpha + i\beta \\ \eta = \alpha - i\beta \end{array}\right\}. \tag{11}$$

From this, we define the new real variables α and β as follows.

$$\left.\begin{array}{l} \alpha = \frac{1}{2}(\xi + \eta) \\ \beta = \frac{1}{2i}(\xi - \eta) \end{array}\right\}. \tag{12}$$

Under the change of variables (12), the equation (2) turns to

$$u_{\alpha\alpha} + u_{\beta\beta} = H_6(\alpha, \beta, u, u_\alpha, u_\beta), \tag{13}$$

which is called *canonical form of the elliptic equation.*

Example 3. Let's consider equation $u_{xx} + x^2 u_{yy} = 0$. The discriminant of this equation with $A = 1,\ B = 0,\ C = x^2$ is $\Delta = B^2 - 4AC = -4x^2 < 0\ ,\quad x \neq 0$. Characteristic equations are obtained from

$$\left(\frac{dy}{dx}\right)^2 + x^2 = 0$$

$$\frac{dy}{dx} = ix \qquad \text{and} \qquad \frac{dy}{dx} = -ix$$

and from the integration of the last equations, we have

$$2y - ix^2 = c_1 \qquad \text{and} \qquad 2y + ix^2 = c_2$$

By using the characteristic coordinates ξ and η, we can write

$$\xi = 2y - ix^2 \qquad \text{and} \qquad \eta = 2y + ix^2$$

and we obtain

$$\begin{aligned} \alpha &= \frac{1}{2}(\xi + \eta) = 2y \\ \beta &= \frac{1}{2i}(\xi - \eta) = -x^2. \end{aligned}$$

If the necessary calculations are done under these change of variables, the canonical form of the given equation is found as

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta} u_\beta.$$