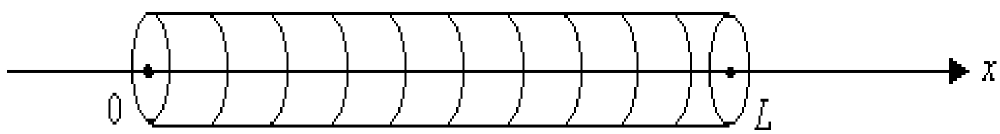
3.3. Solution of Heat equation with initial and boundary conditions, and Uniqueness of solution

Consider a homogeneous straight bar of length L. Let's assume that this bar, which is located along the $0 \leq x \leq L$ range on the x-axis, is thin enough and this situation ensures that the heat distribution over the vertical section of the bar, corresponding to any moment t, can be taken equally.



Heat conduction in a thin bar

Also, let's assume that the lateral surface of this bar is insulated so that there is no heat loss across the surface. In this case, the heat flow through the bar will only be in the x-axis direction. Let us denote by $u(x,t)$ the heat of the vertical section of the bar at point x at any time t. In this case, the function $u(x,t)$, which gives the heat distribution in the bar, will be the solution of the initial and boundary value problem given below.

Initial and Boundary Value Problem

Consider the one-dimensional heat equation

$$u_t - ku_{xx} = 0 \quad ; \qquad 0 < x < L \quad , \qquad t > 0 \tag{1}$$

$$u(x,0) = f(x) \quad ; \qquad 0 \leq x \leq L \tag{2}$$

with initial condition and

$$u(0,t) = 0 \quad , \qquad u(L,t) = 0 \quad ; \qquad t \geq 0 \tag{3}$$

boundary condition.

In the previous section, by the method of separation of variables, we find the solution the initial and boundary value problems of the heat equations described with (1), (2) and (3) in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t} \sin \frac{n\pi x}{L}$$

where the coefficient b_n is given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad ; \qquad n = 1, 2, ...$$

Theorem 1. This solution is unique if

$$\begin{array}{llll} u_t - k u_{xx} = 0 & ; \quad 0 < x < L & , & t > 0 \\ u(x,0) = f(x) & ; \quad 0 \le x \le L & & \\ u(0,t) = 0 & , \quad u(L,t) = 0 & ; & t \ge 0 \end{array}$$

has any solution $u(x,t) \in C^2(0 < x < L) \cap C^1(t > 0)$.

Proof: Let's suppose that, there are two different solutions $u_1(x,t)$ and $u_2(x,t)$. If we denote the difference of u_1 and u_2 with

$$v(x,t) = u_1(x,t) - u_2(x,t),$$

the function $v(x,t)$ will provide homogeneous initial and boundary value problem

$$\left.\begin{array}{llll} v_t - k v_{xx} = 0 & ; \quad 0 < x < L & , & t > 0 \\ v(x,0) = 0 & ; \quad 0 \le x \le L & & \\ v(0,t) = 0 & , \quad v(L,t) = 0 & ; & t \ge 0 \end{array}\right\}. \tag{4}$$

On the other hand, let's define a function w as

$$w(t) = \frac{1}{2k} \int_0^L v^2 \, dx$$

and find the derivative of this function with respect to t as follows.

$$w'(t) = \frac{1}{k} \int_0^L v \, v_t \, dx$$

Considering (4) if we apply the integration by parts, we have

$$w'(t) = \frac{1}{k} \int_0^L v(k \, v_{xx}) \, dx = \int_0^L v \, v_{xx} \, dx = [v \, v_x]_0^L - \int_0^L v_x^2 \, dx.$$

Since

$$v(0,t) = v(L,t) = 0,$$

we obtain

$$w'(t) = -\int_0^L v_x^2 \, dx \le 0. \tag{5}$$

Then

$$w(0) = \frac{1}{2k} \int_0^L v^2(x,0) \, dx = 0 \tag{6}$$

is obtained by the condition $v(x,0)=0$. It is seen from (5) and (6) that $w(t)$ is a non-increasing function of t. That is to say,

$$w(t) \leq 0. \tag{7}$$

But, due to the definition of $w(t)$, we can write

$$w(t) \geq 0. \tag{8}$$

(7) and (8) show that for $t \geq 0$

$$w(t) = 0.$$

For $0 \leq x \leq L$ and $t \geq 0$, since $v(x,t)$ is continuous, we can write

$$\begin{aligned}
w(t) \quad &= \quad \frac{1}{2k}\int_0^L v^2(x,t)\,dx = 0 \\
&\Rightarrow \quad v(x,t) \ = 0 \\
&\Rightarrow \quad v(x,t) \ = u_1(x,t) - u_2(x,t) \equiv 0 \\
&\Rightarrow \quad u_1(x,t) \equiv u_2(x,t)
\end{aligned}$$

so the solution is unique.