5.Wave Equation

5.1.Vibrating string with fixed end points

One of the important applications of partial differential equation is vibrations of elastic strings and membranes. In this section, we consider one dimensional wave equation, representing a uniform vibrating string without external forces,

$$PDE: \ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$BC: \ u(0,t) = 0, \ u(L,t) = 0 \tag{2}$$

$$IC: \ u(x,0) = f(x) \ , \quad u_t(x,0) = g(x) \tag{3}$$

This vibrating problem or wave equation has fixed ends at $x = 0$ and $x = L$ and initial position $f(x)$, and initial velocity, $g(x)$. We apply the method of separation of variables:

$$u(x,t) = \phi(x) H(t) \tag{4}$$

If we substitude (4) into the equation (1), we obtain

$$\phi'' H = c^2 \phi H''$$

or

$$\frac{H''}{c^2 H} = \frac{\phi''}{\phi} = -\lambda.$$

From the homogeneous boundary conditions given by (2), it follows

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The Sturm-Liouville problem becomes

$$\phi'' + \lambda\phi = 0 \quad , \quad \phi(0) = \phi(L) = 0.$$

For $\lambda \leq 0$, it is seen that results in the trivial solution. If we get $\lambda = \beta^2$, then

$$\phi(x) = a_1 \cos(\beta x) + a_2 \sin(\beta x),$$

in which the boundary conditions give

$$a_1 = 0, \ \beta = \frac{n\pi}{L}$$

for nontrivial solutions. The eigenvalues are

$$\beta_n = \frac{n^2\pi^2}{L^2}$$

and the corresponding eigenfunctions are

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

The other differential equation becomes

$$H'' + \frac{n^2\pi^2}{L^2}c^2 H = 0$$

and this equation has the solution

$$H_n(t) = b_1 \cos\left(\frac{n\pi ct}{L}\right) + b_2 \sin\left(\frac{n\pi ct}{L}\right).$$

So, it follows from superposition principle that

$$\begin{aligned} u_n(x,t) &= \phi_n(x) H_n(t) \\ &= \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

The initial position gives that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

where A_n is given by

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

On the other hand,

$$u_t(x,t) = \sum_{n=1}^{\infty}\left[-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

From initial velocity (3), it follows

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

where

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

From trigonometric identities,

$$\begin{aligned}
&\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) \\
= &\ \frac{1}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right)-\frac{1}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right).
\end{aligned}$$

The solution of the one dimensional wave equation can be written as

$$u(x,t)=F(x-ct)+G(x+ct) \quad \text{(D'Alembert's solution)}$$

even if the boundary conditions are not fixed at $x=0$ and $x=L$.

D'Alembert's solution will be obtained in the next section.