5.2. D'Alembert Solution

One of the important applications of partial differential equation is vibrations of elastic strings and membranes. In this section, we consider one dimensional wave equation, representing a uniform vibrating string without external forces,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial conditions

$$u(x,0) = f(x), \tag{2}$$

$$u_t(x,0) = g(x) \tag{3}$$

For a vibrating string with zero displacement at $x = 0$ and $x = L$, we obtained the solution by the method of separation of variables

$$u_n(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right). \tag{4}$$

It is seen that this solution can be written as the sum of a forward-moving wave and backward-moving wave

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds.$$

Initial Value Problem (Infinite Domain)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad ; \quad -\infty < x < \infty \quad , \quad t > 0$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad ; \quad -\infty < x < \infty$$

Since the equation is type of hyperbolic, we apply the substitutions

$$\begin{aligned} \xi &= x + ct, \\ \eta &= x - ct, \end{aligned}$$

then we obtain the canonical form of the equation

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$\Rightarrow \quad u(x,t) = F(\xi) + G(\eta), \quad F, G \in C^2$$

$$\Rightarrow \quad u(x,t) = F(x+ct) + G(x-ct) \tag{5}$$

The condition
$$u(x,0) = f(x)$$
gives
$$f(x) = F(x) + G(x). \qquad (6)$$
On the other hand, from (5), we get
$$u_t(x,t) = c\,F^{'}(x+ct\ ) - c\,G^{'}(x-ct).$$
From the condition
$$u_t(x,0) = g(x),$$
we obtain
$$g(x) = c\,F^{'}(x) - c\,G^{'}(x). \qquad (7)$$
By the equations (6) and (7), it follows
$$u(x,t) = \frac{f(x+ct\ ) + f(x-ct)}{2} + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)\,ds.$$
This formula is called D'Alembert formula.

Example 1. Find the solution of wave equation
$$u_{tt} - c^2 u_{xx} = 0$$
with initial conditions
$$u(x,0) = \sin x \quad , \quad u_t(x,0) = 0 \quad ; \quad -\infty < x < \infty$$

Solution: From D'Alembert formula
$$\begin{aligned} u(x,t) &= \frac{\sin(x+ct\ ) + \sin(x-ct)}{2} \\ &= \frac{\sin x\ \cos ct\ + \cos x\ \sin ct + \sin x\ \cos ct - \cos x\ \sin ct}{2} \\ &= 2\frac{\sin x\ \cos ct}{2} = \sin x\ \cos ct\,, \end{aligned}$$
which is desired. It is easily seen that the solution $u(x,t)$ satisfies both the equation and initial conditions.

Example 2. Find the solution of wave equation
$$u_{tt} - c^2 u_{xx} = 0$$
with initial conditions
$$u(x,0) = 1\ , \quad u_t(x,0) = \sin 2x.$$

Solution: From D'Alembert formula,we have

$$
\begin{aligned}
u(x,t) &= 1+\frac{1}{2c}\int\limits_{x-ct}^{x+ct}\sin 2s\,ds = 1+\frac{1}{2c}(-\frac{1}{2}\cos 2s)\Big|_{x-ct}^{x+ct} \\
&= 1+\frac{1}{4c}\left[\cos 2(x-ct)-\cos 2(x+ct)\right] \\
&= 1+\frac{1}{4c}\left[\cos 2x\,\cos 2ct+\sin 2x\,\sin 2ct-\cos 2x\,\cos 2ct+\sin 2x\,\sin 2ct\right] \\
&= 1+\frac{2\sin 2x\,\sin 2ct}{4c} = 1+\frac{\sin 2x\,\sin 2ct}{2c}.
\end{aligned}
$$