**Wave Equation with initial conditions**

The solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial conditions

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x) \end{aligned} \tag{2}$$

is given by D'Alembert formula

$$u(x,t) = \frac{f(x+ct\ ) + f(x-ct)}{2} + \frac{1}{2c} \int\limits_{x-ct}^{x+ct} g(s)\, ds. \tag{3}$$

**Uniqueness of D'Alembert Solution**

Let's see that the D'Alembert formula (3) we obtained as the solution of the initial value problem for the homogeneous wave equation, or in other words, the D'Alembert solution is only one.

It is easily realized by deriving directly that if $f \in C^2(-\infty,\infty)$ and $g \in C^1(-\infty,\infty)$ are functions with (3) the function $u\,(x,t)$ defined by D'Alembert solution is $C^2$ It is of the $(-\infty,\infty)$ class and realizes equation (1) and (2) initial conditions. On the other hand, the facts in the formation of the D'Alembert formula show that any solution of a problem given by (1) and (2) in the class $C^2(-\infty,\infty)$ must have the representation (3). So when f and g are given, the solution is defined as one. So the D'Alembert solution is the only solution of (1) and (2).

**Continuous Dependence on Initial Data**

If, for any fixed time interval $0 \leq t \leq T < \infty$, the solution $u$ changes small amount when the initial data changes a small amount, then the solution $u$ is said to be continuous dependence on the initial data $f$ and $g$.

**Theorem 1** *The D'Alembert solution of the initial value problem is continuous dependence on the initial data.*

**Proof.** Suppose that the solution of equation (1) corresponding to the initial data $f_1$ and $g_1$ is $u_2$, the solution corresponding to the initial data $u_1$, $f_2$ and $g_2$. In this case, for each given $\varepsilon > 0$ there can be a number $\delta > 0$ such that for each $x \in \mathbb{R}$ and $0 \leq t \leq T$

$$|f_1(x) - f_2(x)| < \delta \quad , \quad |g_1(x) - g_2(x)| < \delta$$

the following is satisfied

$$|u_1(x,t) - u_2(x,t)| < \varepsilon.$$

Indeed, it is seen by the use of D'Alembert solution forms belonging to $u_1$ and $u_2$ that,

$$\begin{aligned} |u_1(x,t) - u_2(x,t)| \quad \leq \quad & \frac{1}{2}\,|f_1(x-ct) - f_2(x-ct)| \\ & + \frac{1}{2}\,|f_1(x+ct) - f_2(x+ct)| \\ & + \frac{1}{2c}\int_{x-ct}^{x+ct} |g_1(s) - g_2(s)|\,ds \end{aligned}$$

$$\begin{aligned} |u_1(x,t) - u_2(x,t)| \quad \leq \quad & \frac{1}{2}(\delta + \delta) + \frac{1}{2c}.2cT.\delta \\ < \quad & \frac{1}{2}2\delta + T\delta \; = (1+T)\delta \end{aligned}$$

If we choose $\delta = \dfrac{\varepsilon}{1+T}$, we have

$$|u_1(x,t) - u_2(x,t)| < (1+T)\frac{\varepsilon}{1+T} = \varepsilon$$

which is desired.

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**Remark 2** *If a problem has unique solution and this solution is continuous dependence on initial or boundary data, that problem is said to be well-posed or well defined. So, the initial value problem given by (1) and (2) is a well posed problem. Obviously, well-posed problems give more meaningful results in appli-*

*cations. Because usually only approximate values of boundary or initial data are known. The corresponding solutions in this case show only one approach to the complete solution of the problem. It can not be said that an initial value problem or boundary value problem dealing with a second order partial differential equation is always well established. For example, although an initial value problem for the hyperbolic wave equation is well-posed, it is sometimes not true for the Laplace Equation of the elliptic type.*