6.3. Properties of Sturm-Liouville problems

Regular Sturm–Liouville Eigenvalue Problem

Let the functions $p(x)$, $q(x)$, and $\sigma(x)$ be real and continuous everywhere (including the endpoints) and let $p(x) > 0$ and $\sigma(x) > 0$ everywhere (also including the endpoints). A regular Sturm–Liouville problem is formed of the Sturm–Liouville differential equation

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \quad , \quad a < x < b \tag{1}$$

with boundary conditions

$$\begin{aligned} \alpha_1\phi(a) + \beta_1\phi'(a) &= 0 \\ \alpha_2\phi(b) + \beta_2\phi'(b) &= 0 \end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real.

Any regular Sturm–Liouville problem has the following properties:

1. All the eigenvalues λ of any regular Sturm–Liouville problem are real.

2. There exist an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < ... < \lambda_n < \lambda_{n+1} < ...$$

a. There is a smallest eigenvalue, here we denote it by λ_1.

b. There is not a largest eigenvalue and $\lim_{n\to\infty} \lambda_n = \infty$.

3. For each eigenvalue λ_n, there is an eigenfunction ϕ_n, which is unique to with an arbitrary multiplicative constant. Also, $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.

4. Eigenfunctions ϕ_n and ϕ_m belonging to different eigenvalues λ_n and λ_m are orthogonal with respect to the weight function $\sigma(x)$ .That is,

$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \quad , \quad \lambda_n \neq \lambda_m.$$

5. The set of eigenfunctions $\phi_n(x)$ is a "complete" set and any piecewise smooth function $g(x)$ can be represented by a generalized Fourier series of these eigenfunctions:

$$g(x) \backsim \sum_{n=1}^{\infty} c_n\phi_n(x)\,.$$

This series converges to $\left[g\left(x+\right)+g\left(x-\right)\right]/2$ for $a<x<b$.

Example and Illustration of these properties

We consider a simplest example of a regular Sturm–Liouville problem as follows:

$$\begin{aligned}\frac{d^2\phi}{dx^2}+\lambda\phi &= 0\\ \phi\left(0\right) &= 0\\ \phi\left(L\right) &= 0\end{aligned}$$

The eigenvalues and corresponding eigenfunctions are as follows

$$\lambda_n=\frac{n^2\pi^2}{L^2}\quad\text{and}\quad\phi_n\left(x\right)=\sin\left(\frac{n\pi x}{L}\right)\quad,\quad n=1,2,...$$

1) All eigenvalues are real.

2) There is an infinite number of eigenvalues for the given reguler Sturm-Liouville problem. The smallest eigenvalue is $\lambda_1=\frac{\pi^2}{L^2}$. But, there is no largest eigenvalue, $\lim\limits_{n\to\infty}\lambda_n=\infty$

3) For the eigenvalue $\lambda_n=\frac{n^2\pi^2}{L^2}$, the corresponding eigenfunction is

$$\phi_n\left(x\right)=\sin\left(\frac{n\pi x}{L}\right)$$

The and the eigenfunction is unique. Also, the eigenfunction $\phi_n\left(x\right)=\sin\left(\frac{n\pi x}{L}\right)$ has exactly $n-1$ zeros. Indeed,

For $n=1$, the eigenfunction is $\phi_1\left(x\right)=\sin\left(\frac{\pi x}{L}\right)$ and it has no zero in $0<x<L$
For $n=2$, the eigenfunction $\phi_2\left(x\right)=\sin\left(\frac{2\pi x}{L}\right)$ has one zero in $0<x<L$

For $n=3$, the eigenfunction $\phi_3\left(x\right)=\sin\left(\frac{3\pi x}{L}\right)$ has two zeros in $0<x<L$ and so on.

4) The eigenfunctions $\phi_n\left(x\right)=\sin\left(\frac{n\pi x}{L}\right)$, $n=1,2,...$ are orthogonal with respect to the weight function $\sigma\left(x\right)=1$. Indeed,

$$\int\limits_0^L\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=0$$

for $m\neq n$.

5) Any piecewise smooth function $g(x)$ can be represented in terms of the eigenfunctions $\phi_n\left(x\right)=\sin\left(\frac{n\pi x}{L}\right)$:

$$g\left(x\right)\backsim\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{L}\right),\tag{2}$$

which is a Fourier sine series. The infinite series converges to

$$\frac{[g\left(x+\right)+g\left(x-\right)]}{2}$$

for $0<x<L$. It converges to $g(x)$ for $0<x<L$ if $g(x)$ is continuous.

If we multiply (2) by $\phi_m\left(x\right)=\sin\left(\frac{m\pi x}{L}\right)$ and then we take integral on the interval $0<x<L$, by use of the orthogonality relation we obtain

$$c_n=\frac{\int\limits_0^L g\left(x\right)\sin\left(\frac{n\pi x}{L}\right)dx}{\int\limits_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx}$$

Since $\int\limits_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx=L/2,$ we have

$$c_n=\frac{2}{L}\int\limits_0^L g\left(x\right)\sin\left(\frac{n\pi x}{L}\right)dx.$$