2.2. Canonical Forms of Equations with Constant Coefficients

Consider the equation given by (1)

$$Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x,y). \qquad (1)$$

If the coefficients in the equation are real constants,

$$\Delta = B^2 - 4AC$$

the discriminant will be constant and the equation is the same type at all points of the region. Thus, for equation (1) characteristic curves satisfying the characteristic equation

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

are the two line families defined by the equations

$$\begin{aligned} y - \lambda_1 x &= c_1, \\ y - \lambda_2 x &= c_2. \end{aligned}$$

Here λ_1 and λ_2 are the roots of the algebraic equation $A\lambda^2 - B\lambda + C = 0$ and, c_1 and c_2 are arbitrary constants. In this case, the characteristic coordinates of ξ and η will be in the form of following characteristic coordinates

$$\begin{aligned} \xi &= y - \lambda_1 x \\ \eta &= y - \lambda_2 x \end{aligned}$$

Let's give an example for different types of such equations.

Example 1. Obtain the canonical form of the equation $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$.

Solution: Since $A = 4$, $B = 5$, $C = 1$, we have $\Delta = B^2 - 4AC = 9 > 0$, the equation is hyperbolic type everywhere. Since the characteristic equations are as follows

$$\frac{dy}{dx} = 1 \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{4}$$

then the characteristic curves are following line families

$$y = x + c_1 \qquad \text{and} \qquad y = \frac{x}{4} + c_2$$

So, under the substitutions

$$\xi = y - x \qquad \text{and} \qquad \eta = y - \frac{1}{4}x.$$

we have the canonical form

$$u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{8}{9}.$$

This is the first canonical form of the hyperbolic type equation. If we apply the substitutions

$$\begin{aligned}\alpha &= \xi+\eta \\ \beta &= \xi-\eta,\end{aligned}$$

we find the second canonical form of the same equation as follows

$$u_{\alpha\alpha}-u_{\beta\beta}=\frac{1}{3}u_\alpha-\frac{1}{3}u_\beta-\frac{8}{9}.$$

Example 2. Obtain the canonical form of the equation $u_{xx}-4u_{xy}+4u_{yy}=e^y$.

Solution: Since $A=1$, $B=-4$, $C=4$, we have $\Delta=B^2-4AC=(-4)^2-4.4=0$ and the equation is the parabolic type everywhere. The characteristic equation is as follows

$$\left(\frac{dy}{dx}\right)^2+4\frac{dy}{dx}+4=\left(\frac{dy}{dx}+2\right)^2=0,$$

from which, characteristic line is found as

$$y+2x=c.$$

Thus by choosing η arbitrarily, when we apply the substitution

$$\xi=y+2x \qquad \text{and} \qquad \eta=y$$

we arrive at the canonical form in the form

$$u_{\eta\eta}=\frac{1}{4}e^\eta.$$

Example 3. Obtain the canonical form of the equation $u_{xx}+u_{xy}+u_{yy}+u_x=0$.

Solution: Since $A=1$, $B=1$, $C=1$ we have $\Delta=B^2-4AC=-3<0$ and we find that the equation is the elliptic type everywhere. The characteristic equation is as follows

$$\left(\frac{dy}{dx}\right)^2-\frac{dy}{dx}+1=0$$

$$\frac{dy}{dx}=\frac{1\mp\sqrt{1-4}}{2}=\frac{1}{2}\mp i\frac{\sqrt{3}}{2}$$

and characteristic curves are found as

$$y-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x=c_1 \qquad \text{and} \qquad y-\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x=c_2$$

Thus

$$\xi = y - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x \qquad \text{and} \qquad \eta = y - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x.$$

By the help of these complex characteristic coordinates, define

$$\alpha = \frac{1}{2}(\xi + \eta) = y - \frac{1}{2}x \qquad \text{and} \qquad \beta = \frac{1}{2i}(\xi - \eta) = -\frac{\sqrt{3}}{2}x.$$

By applying this change of variable, the canonical form of the given partial differential equation is given by

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{2}{3}u_\alpha + \frac{2}{\sqrt{3}}u_\beta.$$