2.2. First Order Linear Partial Differential Equations, Lagrange's Method

Let $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ be continuous differentiable functions with respect to each of the variables. Being x, y independent variables and $z = z(x, y)$ dependent variable, consider

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \tag{1}$$

This equation is called the first order quasi-linear partial differential equation. A method for solving such an equation was first given by Lagrange. For this reason, equation (1) is also called the Lagrange linear equation. If P and Q are independent of z and

$$R(x, y, z) = G(x, y) - C(x, y)z,$$

(1) gives the equation with linear partial differential, so a linear partial differential equation can also be considered as a quasi-linear partial differential equation. Therefore, the Lagrange method is also valid for linear partial differential equations.

Lagrange's Method

Let's assume that in a region of three-dimensional space, the functions P and Q are not both zero, and that the function $z = f(x, y)$ has a solution to the equation (1). Considering a fixed point $M(x, y, z)$ on the S surface defined by $z = f(x, y)$, we can give a simple geometrical meaning to equation (1).

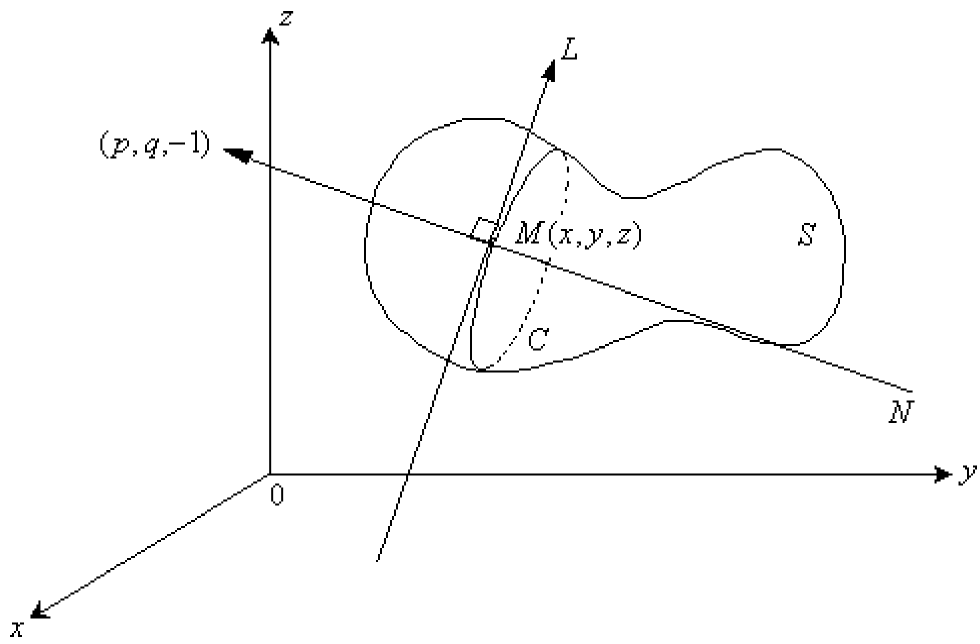

Figure 2.2.1

The normal vector N of surface S at point M is given by

$$\begin{aligned} \vec{n} &= grad\{f(x,y) - z\} \\ &= (f_x, f_y, -1) \\ &= (p, q, -1). \end{aligned}$$

If we write the equation (1) in the form

$$Pp + Qq - R = 0, \tag{2}$$

it is seen that the scalar product of the vectors $(p, q, -1)$ and (P, Q, R) is zero. These two vectors are perpendicular to each other. This means that there is a line L that passes through the point M and is perpendicular to the normal vector n, such that the direction cosines (P, Q, R) of L is tangent to the surface S. Let the plane passing through N and L cut the surface S along a curve C. The direction cosines of the tangent of C on M is (dx, dy, dz) and this tangent is parallel to L. Therefore, the direction cosines of these two lines must be proportional. That is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{3}$$

The first order ordinary differential equation system formed by the equations (3) is called the auxiliary system of the Lagrange equation or the Lagrange system. A system equivalent to system (3), being x independent variable, is

$$\frac{dy}{dx} = \frac{Q}{P} \quad , \quad \frac{dz}{dx} = \frac{R}{P}. \tag{4}$$

The general solution of (4) is

$$y = y(x, c_1, c_2) \quad , \quad z = z(x, c_1, c_2) \tag{5}$$

where c_1 and c_2 are arbitrary constants. If these equations are solved according to c_1 and c_2 , the general solution of the system (3) can be as follows

$$u(x, y, z) = c_1 \ , \ v(x, y, z) = c_2. \tag{6}$$

Each of $u = c_1$ and $v = c_2$ is called first integral of Lagrange system. The functions u and v must also be functionally independent. So at any point $M(x, y, z) \in \Omega$, all Jacobians

$$\frac{\partial(u, v)}{\partial(x, y)} \quad , \quad \frac{\partial(u, v)}{\partial(x, z)} \quad , \quad \frac{\partial(u, v)}{\partial(y, z)}$$

should not be zero at once. Each of the first integrals obtained by (6) is a surface family of one-parameter. Intersection curves of surfaces defined by (6) form the surfaces

$$F(u, v) = 0 \tag{7}$$

The equation (7), where F is an arbitrary function, gives the general solution to the partial differential equation (1).

It is also possible to explain this situation as follows: exact differential of (6) is in the form

$$\left.\begin{aligned} u_x dx + u_y dy + u_z dz = 0 \\ v_x dx + v_y dy + v_z dz = 0 \end{aligned}\right\} \tag{8}$$

Since u and v are the solutions of the system (3), the equations (3) and (8) show that u and v functions satisfy

$$\left.\begin{aligned} u_x P + u_y Q + u_z R = 0 \\ v_x P + v_y Q + v_z R = 0 \end{aligned}\right\}. \tag{9}$$

If we solve the system (9) according to P, Q and R, we obtain

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \tag{10}$$

On the other hand, from the equation $F(u, v) = 0$, we eliminate the arbitrary function F, we obtain the partial differential equation

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}. \tag{11}$$

If the expressions P, Q, R in (10), which are proportional with Jacobians, are written in (11), we have

$$Pp + Qq = R,$$

which shows that (7) is the solution of (1). Since F is arbitrary in this solution, it is the general solution.

Example 1. Find the general solution of the equation $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$.
Solution: The corresponding Lagrange system is in the form

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}.$$

From this system, the first integrals are obtained as follows:

$i)$ From $\frac{dx}{x^2} = \frac{dy}{y^2}$,we have $-\frac{1}{x} = -\frac{1}{y} + c_1$ or $u = \frac{1}{y} - \frac{1}{x} = c_1$

$ii)$ From $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dx - dy}{x^2 - y^2} = \frac{dx - dy}{(x-y)(x+y)} = \frac{dz}{(x+y)z}$ it follows $\frac{d(x-y)}{(x-y)} = \frac{dz}{z}$.

$\Rightarrow \ln(x-y) = \ln z + \ln c_2 \Rightarrow v = \frac{x-y}{z} = c_2.$

So, the general solution of the given equation is

$$F(\frac{1}{y}-\frac{1}{x}\,,\ \ \frac{x-y}{z})=0$$

where F is arbitrary function.

Remark: The general solution given above is also written as

$$z=(x-y)f(\frac{1}{y}-\frac{1}{x})$$

where f is arbitrary function.

Example 2. Find the general solution of the equation $xzp+yzq=-(x^2+y^2)$.

Solution: The corresponding Lagrange system is

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{-(x^2+y^2)}.$$

The first integrals:

$i)\ \frac{dx}{xz}=\frac{dy}{yz}\ \ \Rightarrow\frac{dx}{x}=\frac{dy}{y}\ \ \Rightarrow\ln x-\ln y=\ln c_1\Rightarrow u=\frac{x}{y}=c_1.$

$ii)\ \frac{dx}{xz}=\frac{dy}{yz}=\frac{xdx}{x^2z}=\frac{ydy}{y^2z}=\frac{xdx+ydy}{z(x^2+y^2)}=\frac{dz}{-(x^2+y^2)}$

$\Rightarrow xdx+ydy=-zdz\Rightarrow xdx+ydy+zdz\Rightarrow\ \ v=x^2+y^2+z^2=c_2.$

The general solution of the given equation is

$$F(\frac{x}{y}\,,\ \ x^2+y^2+z^2)=0$$

where F is arbitrary function.

Example 3. Find the general solution of the equation $(y+x)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=\frac{x^2+y^2}{z}..$

Solution: The corresponding Lagrange system is

$$\frac{dy}{x-y} = \frac{dx}{y+x} = \frac{dz}{\dfrac{x^2+y^2}{z}}.$$

The first integrals:

$i)$ From $\dfrac{dy}{x-y} = \dfrac{dx}{y+x} = \dfrac{dx+dy}{2x}$ we have $\dfrac{dx}{y+x} = \dfrac{dx+dy}{2x}$.

$(x+y)d(x+y) - 2xdx = 0 \Rightarrow \; d\left[\dfrac{(x+y)^2}{2} - x^2\right] = 0 \Rightarrow (x+y)^2 - 2x^2 = c_1$ or
$\Rightarrow u(x,y,z) = y^2 + 2xy - x^2 = c_1.$

$ii)$ From $\dfrac{ydy - xdx}{y(x-y) - x(x+y)} = \dfrac{zdz}{x^2+y^2}, \; \dfrac{ydy - xdx}{-(x^2+y^2)} = \dfrac{zdz}{x^2+y^2} \Rightarrow ydy - xdx + zdz = 0$

$\Rightarrow v(x,y,z) = y^2 - x^2 + z^2 = c_2$

The general solution is

$$F(y^2 + 2xy - x^2, \;\; y^2 - x^2 + z^2) = 0$$

where F is arbitrary function.

Remark: It should be noted that the first independent pair of integrals obtained above is not the only pair used to write the general solution. In the last example, the pair of first integrals

$$\begin{aligned} w(x,y,z) &= z^2 + 2xy = c_1 \\ v(x,y,z) &= y^2 - x^2 + z^2 = c_2 \end{aligned}$$

form an independent pair of the first integrals of the auxiliary equation system. Hence the general solution can writen as

$$F(z^2 + 2xy, \;\; y^2 - x^2 + z^2) = 0$$

where F is an arbitrary function.