2.3. Cauchy Problem for First Order Partial Differential Equations

The problem of finding integral curve passing through a certain point of the xy-plane of the ordinary differential equation $y' = f(x, y)$ is one of the main problems in the study of differential equations. Under the fairly general conditions that f must satisfy, the problem has only one solution. A similar problem in studying a partial differential equation of the first order with two independent variables is the problem of finding an integral surface through a given curve in xyz space. This problem is called *the initial value problem* or the *Cauchy problem*.

Solution to Cauchy Problem

We saw in the previous section how to find general solutions of first order linear partial differential equations. Let's see how to use parametric equations to determine an integral surface passing through a Γ curve which is given below

$$\Gamma \, : \; x = x(t) \; , \; y = y(t) \; , \; z = z(t). \tag{1}$$

Consider the equation

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z). \tag{2}$$

The corresponding Lagrange system is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{3}$$

Suppose that two solutions of this system are obtained as follows

$$u(x, y, z) = c_1 \quad , \quad v(x, y, z) = c_2.$$

We have seen in the previous section that all surfaces satisfying the partial differential equation (2) are represented by the equation $F(u, v) = 0$, which will be obtained from a relation $F(c_1, c_2) = 0$ between the arbitrary constants c_1 and c_2. If there is an integral surface passing through the curve Γ between these surfaces, its equation will correspond to a special case of the arbitrary function F. Since the points of the curve Γ are found on this surface, the particular solution must be such that

$$u[x(t), y(t), z(t)] = c_1 \quad , \quad v[x(t), y(t), z(t)] = c_2. \tag{4}$$

Between these two equations in (4), when the parameter t is eliminated, we can obtain

$$\Phi(c_1, c_2) = 0. \tag{5}$$

Thus the solution of the Cauchy problem, i.e., the equation of the integral surface passing through the curve Γ of the partial differential equation (2) can be found

$$\Phi(u, v) = 0. \tag{6}$$

Example 1. Find the solution of the equation $2u_x - 3u_y + 2u = 2x$ such that $u = x^2$ for $y = -\dfrac{x}{2}$.

Solution: The given partial differential equation is linear and we can write its general solution as

$$u(x,y) = (x-1) + e^{-x} f(3x+2y).$$

If we apply initial condition in the general solution where f is an arbitrary function, we see that

$$x^2 = (x-1) + e^{-x} f(2x)$$

or

$$f(2x) = (x^2 - x + 1)e^x$$

Here, if we put $2x = t$, we find $x = \dfrac{t}{2}$ and we can obtain f for the desired solution

$$f(t) = (\frac{t^2}{4} - \frac{t}{2} + 1)e^{\frac{t}{2}}.$$

Thus, we arrive at the desired solution of the problem,

$$u(x,y) = (x-1) + e^{-x}\left[\frac{(3x+2y)^2}{4} - \frac{(3x+2y)}{2} + 1\right] e^{\frac{3x+2y}{2}}$$

or

$$u(x,y) = x - 1 + \left[\frac{(3x+2y)^2}{4} - \frac{3x+2y}{2} + 1\right] e^{\frac{x+2y}{2}}$$

and this solution is unique.

Example 2. In order for the equation $2u_x - 3u_y + 2u = 2x$ to have a solution in the form $u = \varphi(x)$ for $y = -\dfrac{3x}{2}$, show that φ should be given as $\varphi(x) = x - 1 + ke^{-x}$ where k is a constant and find the corresponding solution.

Solution: The general solution of the given partial differential equation is as follows

$$u(x,y) = (x-1) + e^{-x} f(3x+2y).$$

If we want the initial condition to be provided,

$$y = -\frac{3x}{2}\ , \quad \varphi(x) = (x-1) + e^{-x} f(0)$$

is obtained. It is not possible to determine f as we want. For every value of f, $f(0) = k$ is constant. Thus, the initial condition is fulfilled if φ is only

$$\varphi(x) = x - 1 + ke^{-x}.$$

In this case, for $f(0) = k$, we can find the the corresponding solution

$$u(x,y) = (x-1) + e^{-x} f(3x+2y).$$

Here the function f is a semi-arbitrary function that has the value $f(0) = k$. Since an infinite number of functions of this type can be taken, the problem in this case has an infinite number of different solutions.

Example 3. Find the equation of the integral surface of the differential equation

$$2y\,(z-3)\,p + (2x-z)\,q = y\,(2x-3)$$

which passes through the circle $z = 0,\ x^2 + y^2 = 2x$.

Example 4. Find the equation of the integral surface of the differential equation

$$(x-y)\,y^2 p + (y-x)\,x^2 q = \left(x^2 + y^2\right) z$$

which passes through the curve $xz = 1,\ y = 0$.

Example 5. Find the particular integral of the differential equation

$$(2xy-1)\,p + \left(z - 2x^2\right) q = 2\,(x - yz)$$

which passes through the line $x = 1,\ y = 0$.