2.4. Surfaces Perpendicular to a Given Family of Surfaces

In this section we will see how to obtain the systems of surfaces orthogonal to a given system of surfaces. Such families are called *orthogonal* or *orthogonal surface families*. Given the one-parameter surface family defined by equation

$$f(x, y, z) = c \tag{1}$$

in three-dimensional space. We find a system of surfaces which cut each of these given surfaces at right angles. Let us assume that the equation of a surface that intersects perpendicularly each of the surfaces in family (1) is given by

$$z = z(x, y). \tag{2}$$

At any (x, y, z) intersection point, the normal vector

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

of the surface (1) and the normal vector $\left(\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, -1\right)$ of the surface (2) is perpendicular to each other; that is, we can write

$$\frac{\partial f}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial z}{\partial y} - \frac{\partial f}{\partial z} = 0$$

or

$$f_x(x, y, z)p + f_y(x, y, z)q = f_z(x, y, z). \tag{3}$$

The equation is the partial differential equation of surfaces perpendicular to the family of surfaces given by (1) and Lagrange system corresponding to this equation is given by

$$\frac{dx}{f_x(x, y, z)} = \frac{dy}{f_y(x, y, z)} = \frac{dz}{f_z(x, y, z)}.$$

The solutions of a first order quasi linear partial differential equation (3) are surfaces perpendicular to each member of the family (1).

Example 1. Find the surface perpendicularly intersecting the family of surfaces with a parameter given by the equation $(x^2 + y^2)z = c$ and passing through the curve $y^2 = x, z = 0$. Here c is a parameter.

Solution: Let's write the given surface family as

$$f(x, y, z) = (x^2 + y^2)z = c$$

Using f_x, f_y and f_z,

$$f_x = 2xz \quad , \quad f_y = 2yz \quad , \quad f_z = x^2 + y^2$$

The Lagrange system, which corresponds to the partial differential equation of orthogonal surfaces is given by

$$\frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{x^2 + y^2}$$

From this, two independent first integrals are as follows

$$u = \frac{x}{y} = c_1 \quad , \quad v = x^2 + y^2 - 2z^2 = c_2.$$

General equation of surfaces perpendicular to a given family of surfaces are given by

$$F(\frac{x}{y}\, , x^2 + y^2 - 2z^2) = 0$$

or

$$x^2 + y^2 - 2z^2 = g(\frac{x}{y})$$

where F and g are arbitrary functions. To find the special surface that passes through the curve $y^2 = x\,,\ z = 0$, we write the parametric equation of curve as

$$y = t,\ x = t^2\ ,\ z = 0.$$

From this, we obtain

$$\begin{aligned} c_1 &= t,\ \ c_2 = t^2 + t^4 \\ &\Rightarrow\ c_2 = c_1^2 + c_1^4 \end{aligned}$$

Thus, the desired surface has the equation

$$x^2 + y^2 - 2z^2 = \left(\frac{x}{y}\right)^4 + \left(\frac{x}{y}\right)^2.$$

Example 2. Find the surface perpendicularly intersecting the family of surfaces with a parameter given by the equation $z = cxy(x^2+y^2)$. Here c is a parameter.

Solution: Let's write the given surface family as

$$f(x, y, z) = \frac{xy(x^2 + y^2)}{z} = \frac{1}{c}$$

Using f_x, f_y and f_z,

$$f_x = \frac{3x^2y + y^3}{z} \quad , \quad f_y = \frac{3y^2x + x^3}{z} \quad , \quad f_z = -\frac{xy(x^2 + y^2)}{z^2}$$

The Lagrange system, which corresponds to the partial differential equation of orthogonal surfaces is given by

$$\frac{zdx}{3x^2y + y^3} = \frac{zdy}{3y^2x + x^3} = \frac{-z^2dz}{xy(x^2 + y^2)}.$$

From this, we can write

$$\Rightarrow \quad \frac{xzdx + yzdy}{xy\left(3x^2+y^2\right) + xy\left(x^2+3y^2\right)} = \frac{-z^2dz}{xy(x^2+y^2)}$$
$$\Rightarrow \quad \frac{xzdx + yzdy}{xy\left(4x^2+4y^2\right)} = \frac{-z^2dz}{xy(x^2+y^2)}$$
$$\Rightarrow \quad xzdx + yzdy = -4z^2dz$$
$$\Rightarrow \quad xdx + ydy = -4zdz$$
$$\Rightarrow \quad x^2 + y^2 + 4z^2 = c_1 = u\left(x, y, z\right)$$

The second solution is

$$\Rightarrow \quad \frac{zdx + zdy}{\left(x+y\right)^3} = -\frac{zdx - zdy}{\left(x-y\right)^3}$$
$$\Rightarrow \quad \frac{d\left(x+y\right)}{\left(x+y\right)^3} = -\frac{d\left(x-y\right)}{\left(x-y\right)^3}$$
$$\Rightarrow \quad \frac{1}{\left(x+y\right)^2} + \frac{1}{\left(x-y\right)^2} = c_2 = v\left(x, y, z\right).$$

The general equation of surfaces perpendicular to a given family of surfaces are given by

$$F(x^2 + y^2 + 4z^2, \frac{1}{\left(x+y\right)^2} + \frac{1}{\left(x-y\right)^2}) = 0$$

or

$$\frac{1}{\left(x+y\right)^2} + \frac{1}{\left(x-y\right)^2} = g(x^2 + y^2 + 4z^2)$$

where F and g are arbitrary functions.