2.6. Compatible Systems

Let's consider the first order partial differential equation system

$$F(x,y,z,p,q)=0 \quad , \quad G(x,y,z,p,q)=0. \tag{1}$$

If the equations $F(x,y,z,p,q)=0$ and $G(x,y,z,p,q)=0$ have common solutions, the system (1) is called the *compatible system*. If

$$J=\frac{\partial(F,G)}{\partial(p,q)}=F_pG_q-G_pF_q\neq 0 \tag{2}$$

is provided, in that case, the two equations of system (1) are independent from each other, and from these two equations, the expressions p and q can be obtained explicitly as

$$p=p(x,y,z) \quad , \quad q=q(x,y,z) \tag{3}$$

in terms of x,y,z. Therefore, the compatibility of the system (1) is equivalent to integrability of the system of equations (3). That is to say

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$$

or in other words the equation

$$p(x,y,z)dx+q(x,y,z)\,dy-dz=0 \tag{4}$$

must be a exact differential that can be integrated. On the other hand, let's remember that the necessary and sufficient condition for a differential expression in the form of

$$P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$$

to be exact differential, i.e. to be integrable, is to satisfy identity which is given below.

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)\equiv 0.$$

In this identity,

$$P=p \quad , \quad Q=q \quad , \quad R=-1$$

is taken, i order to integrate (4),

$$p\frac{\partial q}{\partial z}-q\frac{\partial p}{\partial z}-\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)=0 \tag{5}$$

must be satisfied. Now, considering that p and q are functions of x,y,z, if we take derivatives from two equations of (1) with respect to x, we get the equations

$$F_x+F_p\frac{\partial p}{\partial x}+F_q\frac{\partial q}{\partial x}=0 \quad , \quad G_x+G_p\frac{\partial p}{\partial x}+G_q\frac{\partial q}{\partial x}=0, \tag{6}$$

if we take derivatives with respect to y, we have

$$F_y + F_p \frac{\partial p}{\partial y} + F_q \frac{\partial q}{\partial y} = 0 \quad , \quad G_y + G_p \frac{\partial p}{\partial y} + G_q \frac{\partial q}{\partial y} = 0. \tag{7}$$

If we take derivatives with respect to z, we obtain

$$F_z + F_p \frac{\partial p}{\partial z} + F_q \frac{\partial q}{\partial z} = 0 \quad , \quad G_z + G_p \frac{\partial p}{\partial z} + G_q \frac{\partial q}{\partial z} = 0. \tag{8}$$

If solving $\frac{\partial q}{\partial x}$ from (6), considering that $J \neq 0$, we have

$$\frac{\partial q}{\partial x} = \frac{F_x G_p - G_x F_p}{J}. \tag{9}$$

Similarly, if $\frac{\partial p}{\partial y}$ from (7) is solved, we can write

$$\frac{\partial p}{\partial y} = \frac{F_q G_y - G_q F_y}{J}. \tag{10}$$

Finally, if $\frac{\partial p}{\partial z}$ and $\frac{\partial q}{\partial z}$ are solved from the two equations of (8), we get

$$\frac{\partial p}{\partial z} = \frac{F_q G_z - G_q F_z}{J} \quad , \quad \frac{\partial q}{\partial z} = \frac{F_z G_p - G_z F_p}{J}. \tag{11}$$

If the expressions (9), (10) and (11) are replaced in (5),

$$p\left(F_z G_p - G_z F_p\right) - q\left(F_q G_z - G_q F_z\right) - \left(F_q G_y - G_q F_y\right) + \left(F_x G_p - G_x F_p\right) \equiv 0 \tag{12}$$

is obtained and this statement is the compatibility condition of the (1) system. The left side of the expression (12) can be shown as

$$[F, G] \equiv \frac{\partial(F,G)}{\partial(x,p)} + p\frac{\partial(F,G)}{\partial(z,p)} + \frac{\partial(F,G)}{\partial(y,q)} + q\frac{\partial(F,G)}{\partial(z,q)} \tag{13}$$

The expression $[F, G]$ is called the crochet of F and G. Therefore, if the system (1) is a compatible system, the crochet of F and G must be equal to zero. That is, the condition

$$[F, G] \equiv 0$$

is the compatibility condition.

Example 1. Show that the equations

$$xp = yq \ , \quad \left(1 + x^2 y^2\right)(xp + yq) = 2xyz$$

are compatible and solve them.

Solution: In the given example, we take

$$F=\left(1+x^2y^2\right)(xp+yq)-2xyz=0 \quad , \quad G=xp-yq=0\,.$$

Since

$$\begin{aligned}
\frac{\partial(F,G)}{\partial(x,p)} &= \begin{vmatrix} F_x & G_x \\ F_p & G_p \end{vmatrix} = 2x^2y^2\,(px+qy)-2xyz \\
\frac{\partial(F,G)}{\partial(z,p)} &= \begin{vmatrix} F_z & G_z \\ F_p & G_p \end{vmatrix} = -2x^2y \\
\frac{\partial(F,G)}{\partial(y,q)} &= \begin{vmatrix} F_y & G_y \\ F_q & Gq \end{vmatrix} = -2x^2y^2\,(px+qy)+2xyz \\
\frac{\partial(F,G)}{\partial(z,q)} &= \begin{vmatrix} F_z & G_z \\ F_q & G_q \end{vmatrix} = 2xy^2,
\end{aligned}$$

we can write

$$\begin{aligned}
[F,G] &= \frac{\partial(F,G)}{\partial(x,p)}+p\frac{\partial(F,G)}{\partial(z,p)}+\frac{\partial(F,G)}{\partial(y,q)}+q\frac{\partial(F,G)}{\partial(z,q)} \\
&= 2x^2y^2\,(px+qy)-2xyz+p\left(-2x^2y\right) \\
&\quad -2x^2y^2\,(px+qy)+2xyz+q\left(2xy^2\right) \\
&= 2xy(yq-xp) \\
&\equiv 0,
\end{aligned}$$

which says that the system is compatible. If we solve the system according to p and q, we have

$$\left.\begin{array}{r} xp-yq=0 \\ xp+yq=\dfrac{2xyz}{1+x^2y^2} \end{array}\right\} \quad \Rightarrow \quad p=\frac{yz}{1+x^2y^2} \quad , \quad q=\frac{xz}{1+x^2y^2}$$

and we obtain

$$\begin{aligned}
dz &= pdx+qdy=\frac{yz}{1+x^2y^2}\,dx+\frac{xz}{1+x^2y^2}\,dy \\
&\Rightarrow \quad \frac{dz}{z}=\frac{ydx+xdy}{1+(xy)^2}=\frac{d(xy)}{1+(xy)^2}
\end{aligned}$$

By integrating this last expression, common solution is found

$$\begin{aligned}
\frac{dz}{z} &= \frac{d(xy)}{1+(xy)^2} \quad \Rightarrow \quad \ln z=\arctan(xy)+\ln c \\
&\Rightarrow \quad z=ce^{\arctan(xy)}
\end{aligned}$$

Here c is an arbitrary constant.