2.8. First-order special type partial differential equations

In this section, we will examine the special types of first-order partial differential equations that can be easily solved with the Charpit method.

A. Equations containing only p and q:

Consider the type equations which do not explicitly include x, y, z variables, but only partial derivatives $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$

$$F(p, q) = 0. \tag{1}$$

For an equation of type (1), we can write

$$F_x = F_y = F_z = 0,$$

so we can see that the corresponding Charpit auxiliary equations are given by

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{0} = \frac{dq}{0}.$$

From the last two equations, we have $dp = 0$ and $dq = 0$. Using $dp = 0$,we have

$$p = a \tag{2}$$

The corresponding value of q is from (1) and (2), so we get

$$F(a, q) = 0 \Rightarrow \quad q = Q(a)$$

Thus an complete integral of equation (1) can be found by integrating

$$dz = pdx + qdy = adx + Q(a)dy$$

and we obtain

$$z = ax + Q(a)y + b. \tag{3}$$

This complete integral with a and b constants indicate a family of planes.

Remark: Starting from $dq = 0$, a solution can be obtained by taking $q = a$.

Example 1. Find a complete integral of the equation $pq^2 - q^3 + \sin q = 0$.

Solution: The given partial differential equation is of a special type with only p and q. For this equation which is first degree with respect to p and third degree with respect to q, it is more appropriate to take

$$\text{from } dq = 0 \Rightarrow \quad q = a.$$

In this case, the value of p will be

$$pa^2 - a^3 + \sin a = 0 \quad \Rightarrow \quad p = \frac{a^3 - \sin a}{a^2}.$$

By integrating

$$dz = pdx + qdy = \frac{a^3 - \sin a}{a^2} dx + ady,$$

we obtain

$$z = \left(\frac{a^3 - \sin a}{a^2} \right) x + ay + b$$

where a and b are arbitrary constants.

B. Equations without independent variables:

For such partial differential equations expressed as

$$F(z, p, q) = 0, \tag{4}$$

we can write $F_x = 0$ ve $F_y = 0$ so that the corresponding Charpit auxiliary equations are in the form of

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-pF_z} = \frac{dq}{-qF_z}$$

From the last two equations, the first integral is given by

$$p = aq \tag{5}$$

After solving p and q in terms of z from (4) and (5) and substituting $dz = pdx + qdy$, the equation is integrated and we obtain a two-parameter complete integral of (4).

Example 2. Find a complete integral of the equation $z = p^2 - q^2$.

Solution: The given partial differential equation is a special type equation which does not contain independent variables, and a first integral from the corresponding Charpit auxiliary equations is $p = aq$. If p and q are solved from this first integral equation and the given partial differential equation, we have

$$\left.\begin{array}{c} z = p^2 - q^2 \\ p = aq \end{array}\right\} \quad \Rightarrow \quad q = \pm\sqrt{\frac{z}{a^2 - 1}} \quad , \quad p = \pm a\sqrt{\frac{z}{a^2 - 1}}$$

$$dz = pdx + qdy = \pm\sqrt{\frac{z}{a^2 - 1}}(adx + dy) \ , \quad \sqrt{\frac{a^2 - 1}{z}} dz = \pm (adx + dy) \, .$$

By the integration of both sides, we obtain complete integral as follows

$$\left(2\sqrt{z\left(a^2 - 1\right)} + b \right)^2 = (ax + y)^2 \, .$$

C. Equations that can be separated by their variables:

If a first-order partial differential equation can be written as

$$f(x,p) = g(y,q) \tag{6}$$

it is said to be of the separable type to its variables. If we write equation (6) as

$$F(x,y,z,p,q) = f(x,p) - g(y,q) = 0,$$

we have

$$F_x = f_x \quad , \quad F_y = -g_y \quad , \quad F_z = 0 \quad , \quad F_p = f_p \quad , \quad F_q = -g_q$$

for the corresponding Charpit auxiliary equations expressed as follows

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{g_y}.$$

We obtain

$$f_x dx + f_p dp = 0 \tag{7}$$

from the first and fourth equations. Since the function f depends only on x and p, the expression (7) is an ordinary exact differential equation with respect to x and p. With the help of the

$$df(x,p) = f_x dx + f_p dp = 0$$

we have the solution to (7)

$$f(x,p) = a \tag{8}$$

where a is an arbitrary constant. From (6) and (8), we have

$$g(y,q) = a. \tag{9}$$

Thus, from (8) and (9), we obtain

$$p = p(x,a) \qquad \text{and} \qquad q = q(y,a),$$

respectively. By putting them in

$$dz = p(x,a)dx + q(y,a)dy$$

and integrating the last equality, we obtain the complete integral of equation (6) as follows

$$z = \int p(x,a)dx + \int q(y,a)dy + b.$$

Example 3. Find a complete integral of the equation $p^2y(1+x^2) = qx^2$.

Solution: Since the given partial differential equation can be written as

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y},$$

it is a special type that can be separated into variables. Choosing

$$f(x,p)=\frac{p^2(1+x^2)}{x^2}=a^2 \quad , \quad g(y,q)=\frac{q}{y}=a^2,$$

we have

$$p=\frac{ax}{\sqrt{1+x^2}} \quad , \quad q=a^2y$$

and

$$dz=pdx+qdy=\frac{ax}{\sqrt{1+x^2}}dx+a^2ydy.$$

By integrating last expression, the desired complete integral is obtained as follows.

$$z=a\sqrt{1+x^2}+\frac{1}{2}a^2y^2+b.$$

D. Clairaut Equation

If a first order partial differential equation can be expressed as

$$z=xp+yq+f(p,q), \tag{10}$$

it is called the *Clairaut type equation*. If we write the equation (10) as

$$F(x,y,z,p,q)=xp+yq+f(p,q)-z=0$$

we have corresponding Charpit auxiliary equations as

$$F_x=p \quad , \quad F_y=q \quad , \quad F_z=-1 \quad , \quad F_p=x+f_p \quad , \quad F_q=y+f_q,$$

$$\frac{dx}{x+f_p}=\frac{dy}{y+f_q}=\frac{dz}{px+qy+pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}.$$

From the last two equations, we find

$$dp=0\Rightarrow p=a \quad \text{and} \quad dq=0\Rightarrow q=b$$

and If these values of p and q are replaced in (10), the complete integral

$$z=ax+by+f(a,b) \tag{11}$$

is obtained. A partial differential equation of type (10) usually also has a singular integral. This singular integral is the envelope of the family of planes defined by (11).

Example 4. Find a complete integral $pqz=p^2\left(xq+p^2\right)+q^2\left(yp+q^2\right)$.

Solution: Since the given partial differential equation can be written as

$$\begin{aligned} z &= \frac{p}{q}\left(xq+p^2\right)+\frac{q}{p}\left(yp+q^2\right) \\ &\Rightarrow \; z=xp+yq+\frac{p^3}{q}+\frac{q^3}{p}, \end{aligned}$$

it is a Clairaut type equation. By inserting $p = a$, $q = b$, we have the complete integral

$$z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}.$$