2.9. Partial differential equations transformed special type equations

Most of the nonlinear first order partial differential equations will naturally not be included in the special type equations class given in the previous section, so their solution by the Charpit method will not be easy. However, there are some equations that can be reduced to special types by appropriate transformations. Let's see these types of equations below.

I. Equations of type

$$F(x^m p, y^n q) = 0 \tag{1}$$

with m and n being any real constants:

If change of variable $x_1 = x^{\,1-m}$ is applied in (1) for $m \neq 1$, we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dx} = (1-m)x^{-m}\frac{\partial z}{\partial x_1}$$

or

$$x^m p = (1-m)\frac{\partial z}{\partial x_1} = (1-m)p_1.$$

If $m = 1$, then the change of variable $x_1 = lnx$ is done and in this case, we can write

$$xp = \frac{\partial z}{\partial x_1} = p_1.$$

Similarly, If we say $y_1 = y^{1-n}$ for $n \neq 1$, we obtain

$$y^n q = (1-n)\frac{\partial z}{\partial y_1} = (1-n)q_1$$

and also If $n = 1$, we can apply the substitution $y_1 = \ln y$ and we can write

$$yq = \frac{\partial z}{\partial y_1} = q_1.$$

Thus, in each case of m and n, the equation (1) can always be obtained in the form of

$$F_1(p_1, q_1) = 0. \tag{2}$$

The equation (2) is of special type which contains only derivatives $p_1 = \dfrac{\partial z}{\partial x_1}$ and $q_1 = \dfrac{\partial z}{\partial y_1}$ and how the solution was obtained was seen in (A) of the previous part. After obtaining a complete integral of equation (2) dependent on x_1, y_1 and z, substituting the values given by transformations in terms of x and y for x_1 and y_1, a complete integral of equation (1) is obtained.

II. Equations of type

$$F(z^k p, z^k q) = 0 \tag{3}$$

where k is any real constant:

If we say $z_1 = z^{k+1}$ with $k \neq -1$, we can write

$$p_1 = \frac{\partial z_1}{\partial x} = \frac{dz_1}{dz}\frac{\partial z}{\partial x} = (k+1)z^k p, \tag{4}$$

$$q_1 = \frac{\partial z_1}{\partial y} = \frac{dz_1}{dz}\frac{\partial z}{\partial y} = (k+1)z^k q. \tag{5}$$

Subtracting the values $z^k p$ and $z^k q$ from (4) and (5) and replacing them in (3), we obtain

$$F_1(p_1, q_1) = 0,$$

which is the special type equation containing only derivatives of p_1 and q_1. After solving this equation by known methods, the complete integral of equation (3) is found by substituting z^{k+1} for z_1.

If we put $k = 1$ in equation (3) and then we apply change of variable $z_1 = lnz$, it follows

$$p_1 = \frac{\partial z_1}{\partial x} = \frac{p}{z}, \quad q_1 = \frac{\partial z_1}{\partial y} = \frac{q}{z}$$

and the equation (3) is again reduced to the form $F_1(p_1, q_1) = 0$, which is the special type of the previous part (A).

III. Equations of type

$$F(x^m z^k p,\, y^n z^k q) = 0 \tag{6}$$

with m, n, k are any real constants:

To solve the equations of type (6), it will be sufficient to apply both the transformations applied in (I) and (II) consecutively.

IV. Equations of type

$$F(z,\, x^m p,\, y^n q) = 0 \tag{7}$$

with m and n being any real constants:

For this type of equations, we apply for the transformations which we use in (I) and we obtain

$$F_1(z, p_1, q_1) = 0 \tag{8}$$

V. Equations of type

$$f(x, z^k p) = g(y, z^k q) \tag{9}$$

where k is any real constant:

Such an equation can be converted to a special type which can be separated into variables

$$f_1(x,p_1)=g_1(y,q_1)$$

by applying the transformation $z_1=z^{k+1}$ $(k\neq -1)$ or $z_1=\ln z$ $(k=-1)$ and from which, it is solved by the known method.

Example 1. Find a complete integral of the equation

$$2(y^3+y^2)\left[(1+x^4)p+xq+xz^3\right]+xz^3(3y^2+2y)=0.$$

Solution: If we multiply the given equation by z^{-3}, we can write it as follows

$$2(y^3+y^2)\left[(1+x^4)pz^{-3}+xqz^{-3}+x\right]+x(3y^2+2y)=0.$$

This equation includes both $z^{-3}p$ and $z^{-3}q$, so it would be appropriate to apply the substitution $z_1=z^{k+1}=z^{-3+1}=z^{-2}$. In this case, we have

$$p_1=\frac{\partial z_1}{\partial x}=\frac{dz_1}{dz}\frac{\partial z}{\partial x}=-2z^{-3}p\Rightarrow z^{-3}p=-\frac{1}{2}p_1,$$
$$q_1=\frac{\partial z_1}{\partial y}=\frac{dz_1}{dz}\frac{\partial z}{\partial y}=-2z^{-3}q\Rightarrow z^{-3}q=-\frac{1}{2}q_1.$$

If they are put in place in the equation, we obtain the equation separable to variables as

$$2(y^3+y^2)\left[(1+x^4)(-\frac{1}{2}p_1)+x(-\frac{1}{2}q_1)+x\right]+x(3y^2+2y)=0$$

or

$$\Rightarrow\quad (1+x^4)p_1+xq_1-2x=\frac{x(3y^2+2y)}{y^3+y^2}$$

$$\Rightarrow\quad \frac{(1+x^4)p_1-2x}{x}=\frac{3y^2+2y}{y^3+y^2}-q_1.$$

This equation is of special type $f(x,p_1)=g(y,q_1)$ and it has the first integral which is the form of

$$f(x,p_1)=g(y,q_1)=a.$$

In that case, we have

$$\begin{aligned} f(x,p_1) &= \frac{(1+x^4)p_1-2x}{x}=a \quad , \qquad p_1=\frac{(a+2)x}{1+x^4}, \\ g(y,q_1) &= \frac{3y^2+2y}{y^3+y^2}-q_1=a \quad , \qquad q_1=\frac{3y^2+2y}{y^3+y^2}-a. \end{aligned}$$

Using

$$dz_1 = \frac{\partial z_1}{\partial x} dx + \frac{\partial z_1}{\partial y} dy = p_1 dx + q_1 dy,$$

we write

$$dz_1 = \frac{(a+2)x}{1+x^4} dx + \left(\frac{3y^2+2y}{y^3+y^2} - a \right) dy,$$

and integrating both sides of last expression, we have

$$z_1 = \frac{a+2}{2} \arctan x^2 + \ln(y^3+y^2) - ay + b.$$

Considering that $z_1 = z^{-2}$, a complete integral of the given equation is found as follows

$$z = \left\{ \frac{a+2}{2} \arctan x^2 + \ln(y^3+y^2) - ay + b \right\}^{-1/2}.$$