3.1. Second Order Linear Partial Differential Equations With Constant Coefficients

General form of second order linear partial differential equations with two independent variables x and y is given

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1}$$

where the coefficients A, B, C, D, E, F and G are functions of x and y. Except for some special cases, it is not always possible to obtain the general solution of (1) as in the first order linear partial differential equations.

As a special case, we will consider equation

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \tag{2}$$

where a, b, c, d, e and f are constants, all a, b and c must not be zero simultaneously. In terms of partial derivative operators

$$D_x = \frac{\partial}{\partial x} \quad , \quad D_y = \frac{\partial}{\partial y}$$

by using linear operator

$$L = aD_x^2 + bD_xD_y + cD_y^2 + dD_x + eD_y + f$$

we can write (2) as

$$L(u) = 0 \tag{3}$$

In (3), let's consider the polynomial

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$

obtained by replacing D_x and D_y with x and y, respectively and suppose that the polynomial $P(x, y)$ can be factored out as

$$P(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2).$$

In this case, taking into the consideration L_1 and L_2

$$L_1 = a_1D_x + b_1D_y + c_1 \quad , \quad L_2 = a_2D_x + b_2D_y + c_2 \tag{4}$$

the operator L can be written as factored as

$$L = L_1.L_2$$

Since all the coefficients of L_1 and L_2 are constants, we can write

$$L_1.L_2 = L_2.L_1$$

so we have

$$L = L_1.L_2 = L_2.L_1 \tag{5}$$

In such a case, the L operator is said to be reducible.

If L is reducible, the general solution of equation (3) containing two arbitrary functions can be obtained explicitly in terms of solutions of the first order equations

$$L_1(u) = 0 \quad \text{and} \quad L_2(u) = 0.$$

Let L_1 and L_2 be given as in (4), with $L_1 \neq L_2$. Let the general solution of $L_1(u) = 0$ be $u = u_1$ and the general solution of $L_2(u) = 0$ be $u = u_2$. If $u = u_1 + u_2$, because of the linearity of L and (5), we can write

$$\begin{aligned} L(u) &= L(u_1 + u_2) = L(u_1) + L(u_2) \\ &= L_2 L_1(u_1) + L_1 L_2(u_2) = 0 \end{aligned} \tag{6}$$

This shows that the function $u = u_1 + u_2$ implements equation (3).

On the other hand,

$$L_1(u_1) = (a_1 D_x + b_1 D_y + c_1)\, u_1 = a_1 \frac{\partial u_1}{\partial x} + b_1 \frac{\partial u_1}{\partial y} + c_1 u_1 = 0 \tag{7}$$

Lagrange system corresponding to the first order linear partial differential equation is given by

$$\frac{dx}{a_1} = \frac{dy}{b_1} = \frac{du_1}{-c_1 u_1}$$

From the first two equations of this system, we obtain

$$b_1 x - a_1 y = a$$

If $a_1 \neq 0$, from the first and third equations of the system, we have

$$\ln u_1 = -\frac{c_1}{a_1} x + \ln b \quad \text{or} \quad u_1 = b \exp(-\frac{c_1}{a_1} x)$$

Here, the general solution of (7) is found by inserting $b = f_1\,(a)$,

$$u_1(x, y) = e^{-\frac{c_1}{a_1} x} f_1(b_1 x - a_1 y) \tag{8}$$

where f_1 is an arbitrary function.

If $a_1 = 0$ in (7) then $b_1 \neq 0$ since both a_1 and b_1 cannot be zero simultaneously. In this case, the solution (8) is replaced by

$$u_1(x, y) = e^{-\frac{c_1}{b_1} y} f_1(x) \tag{9}$$

Similarly, if $a_2 \neq 0$, the solution of $L_2(u_2) = 0$ is

$$u_2(x, y) = e^{-\frac{c_2}{a_2} x} f_2(b_2 x - a_2 y) \tag{10}$$

and If $a_2 = 0$ (since $b_2 \neq 0$), we get

$$u_2(x, y) = e^{-\frac{c_2}{b_2} y} f_2(x) \tag{11}$$

where f_2 is an arbitrary function.

Thus, if $a_1 a_2 \neq 0$, the general solution of (3) is given by

$$u(x,y) = e^{-\frac{c_1}{a_1}x} f_1(b_1 x - a_1 y) + e^{-\frac{c_2}{a_2}x} f_2(b_2 x - a_2 y). \tag{12}$$

If $a_1 = 0$ or $a_2 = 0$, the terms corresponding to u_1 and u_2 in (12) will be replaced by (9) or (11), respectively.

To obtain the general solution of the non-homogeneous partial differential equation $L(u) = G$, it is sufficient to find any particular solution of the non-homogeneous equation and add it to the solution (12).

Example 1. Find the general solution of the equation

$$u_{xx} - u_{yy} = 5\cos(2x+y) - 3\sin(2x+y).$$

Solution: The linear operator

$$L = D_x^2 - D_y^2$$

for the given equation can be written as the product of L_1 and L_2,

$$L_1 = D_x - D_y \quad \text{and} \quad L_2 = D_x + D_y.$$

For the operator L_1

$$L_1(u) = (D_x - D_y)\, u = 0,$$

we have general solution of first order homogen linear partial differential equation $L_1(u) = 0$ as follows

$$u_1(x,y) = f(x+y)$$

and for the operator L_2, general solution of the equation

$$L_2(u) = (D_x + D_y)\, u = 0$$

is given by

$$u_2(x,y) = g(x-y).$$

Thus, the general solution of the homogeneous equation $L(u) = 0$ corresponding to the given equation is found in the form

$$u_h = f(x+y) + g(x-y)$$

where f and g are arbitrary twice differentiable functions.

Now let's find a particular solution $u_p(x,y)$ for the non-homogeneous equation

$$L(u) = 5\cos(2x+y) - 3\sin(2x+y).$$

Let's use the method of undetermined coefficients for this equation and let's look for a particular solution in the form

$$u_p(x, y) = A\cos(2x + y) + B\sin(2x + y).$$

If we differentiate this function with respect to x and y, and then we write in the given equation, by comparing similar terms we determine the coefficients A and B as

$$A = -\frac{5}{3}, \ B = 1.$$

Thus, the general solution u of the given partial differential equation can be written

$$u = u_h + u_p$$

$$u = f(x + y) + g(x - y) - \frac{5}{3}\cos(2x + y) + \sin(2x + y)$$

where f and g are arbitrary twice differentiable functions.

Finding the solution for $L_1 = L_2$:

Now let's consider the case $L_1 = L_2$, that is, the L operator is composed of repeated factors. In this case, at least one of a_1 and b_1 is nonzero, we have

$$L_1 = L_2 = a_1 D_x + b_1 D_y + c_1$$

and we need to find the general solution to the equation.

$$L(u) = L_1^2(u) = L_1\left[L_1(u)\right] = \left(a_1 D_x + b_1 D_y + c_1\right)^2 u = 0. \tag{13}$$

Let's assume $a_1 \neq 0$. If $L_1(u) = w$, for u to have a solution of equation (13), w must satisfy the following equation

$$L_1(w) = a_1 w_x + b_1 w_y + c_1 w = 0. \tag{14}$$

Since $a_1 \neq 0$, the general solution of (14) can be written as

$$w(x, y) = e^{-\frac{c_1}{a_1}x} g_1(b_1 x - a_1 y)$$

where g_1 is an arbitrary function. To get the solution u, we have to solve the following equation

$$a_1 u_x + b_1 u_y + c_1 u = e^{-\frac{c_1}{a_1}x} g_1(b_1 x - a_1 y). \tag{15}$$

For this, if we use the Lagrange method given in Section 2.2, the general solution of equation (13) is found as

$$u(x, y) = e^{-\frac{c_1}{a_1}x}\left[x f_1(b_1 x - a_1 y) + f_2(b_1 x - a_1 y)\right] \tag{16}$$

where f_1 and f_2 are arbitrary twice differentiable functions.

Note: The above method is also used to find a particular solution of the non-homogeneous equation $L(u) = G$ if L is reducible. Indeed, with $L = L_1 L_2$, let us assume that $L_1(v) = G$ has a particular solution v and $L_2(u) = v$ has a particular solution u.Since

$$L(u) = L_1 L_2(u) = L_1 \left[L_2(u) \right] = L_1(v) = G$$

Then u turns out that a particular solution of $L(u) = G$. Here L_1 and L_2 need not be different. They can be different or equal.