WEEK 7

7. Continuous Probability Distributions

In this part of the class we are going to review e few special continuous probability distributions (Uniform, Gamma, exponential, chi-square, Weibull, Beta, log normal and normal). First we have to mention that there are so many continuous distributions in the literature. In this part of the class, we are going to give the probability function of the distribution and calculate the mean and the variance. We will also calculate some probabilities.

1. <u>Continuous Uniform Distribution</u>:

We say that a continuous random variable X is uniformly distributed over the range (a,b) if it has a probablity density function

$$f(x)=\begin{cases} \dfrac{1}{b-a} & , \quad a<x<b \\ 0 & , \quad elsewhere. \end{cases}$$

The graphs probability function and distribution function of uniform distribution are given below:

Example: $X \sim U(2,5)$ then the probability density function is

$$f(x)=\begin{cases} \dfrac{1}{3} & , \quad 2<x<5 \\ 0 & , \quad elsewhere. \end{cases}$$

and

$$P(3<X<4)=\int_{x=3}^{4} f(x)\,dx=\int_{x=3}^{4}\frac{1}{3}dx=\frac{1}{3}.$$

The moments of uniform distribution is calculated as follows. If $X \sim U(a,b)$ then the first moment is

$$E(X) = \int_{x=a}^{b} x\, f(x)dx = \frac{1}{b-a}\int_{x=a}^{b} x\, dx = \frac{1}{b-a}\frac{x^2}{2}\Bigg|_{x=a}^{b} = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

and the second moment

$$E(X^2) = \int_{x=a}^{b} x^2 f(x)dx = \frac{1}{b-a}\int_{x=a}^{b} x^2\, dx = \frac{1}{b-a}\frac{x^3}{3}\Bigg|_{x=a}^{b} = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+a^2+ab)}{3(b-a)}$$

$$= \frac{(b^2+a^2+ab)}{3}$$

and therefore the variance of the uniform distribution is calculated as

$$Var(X) = E(X^2)-(E(X))^2 = \frac{(b^2+a^2+ab)}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b^2+a^2+ab)}{3} - \frac{b^2+a^2+2ab}{4} = \frac{(b-a)^2}{12}.$$

We can also calculate the moment generating function of Uniform distribution, but it does not exist at the origine. That is, even the integral exist the function does not generate the moments.

2. Gamma Uniform Distribution:

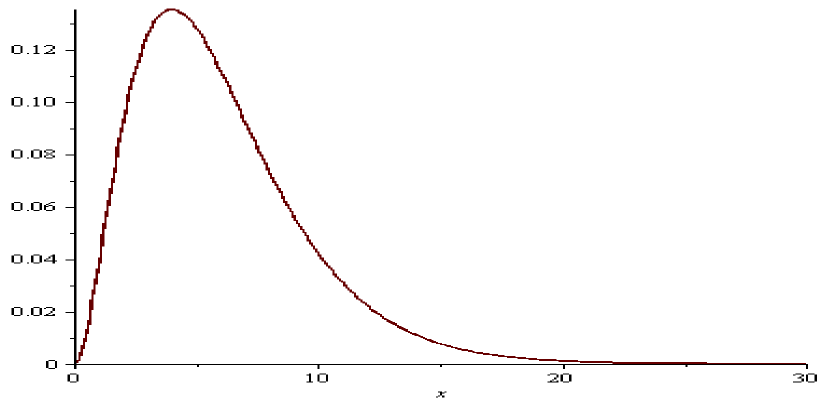
Before we introduce the Gamma distribution let us remember a few properties of the gamma function which you have seen in any calculas class:

1. $\Gamma(\alpha) = \int_{x=0}^{\infty} x^{\alpha-1}e^{-x}dx$ 2. $\Gamma(\alpha)\beta^{\alpha} = \int_{x=0}^{\infty} x^{\alpha-1}e^{-x/\beta}\, dx$
3. $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$ 4. $\Gamma(n+1) = n!$ 5. $\Gamma(1/2) = \sqrt{\pi}$.

Using these properties, we say that a continuous random variable X is distributed as Gamma with parameters α and β and denoted by $X \sim Gamma(\alpha,\beta)$ if it has a probability density function

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1}e^{-x/\beta} & , \quad x > 0 \\ 0 & , \quad elsewhere. \end{cases}$$

The graph of the probability density function of the Gamma distribution is given below:



The probability density function of Gamma Distribution ($\alpha = 3$, $\beta = 2$).

Using the properties of the Gamma function, it is easy to see that the function is a probability density function. Simply,

$$\int_{x=0}^{\infty} f(x)dx = \int_{x=0}^{\infty} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{x=0}^{\infty} x^{\alpha-1} e^{-x/\beta} dx = \frac{\beta^{\alpha}\Gamma(\alpha)}{\beta^{\alpha}\Gamma(\alpha)} = 1.$$

Example: If $X \sim Gamma(2,3)$ then the probability density function is

$$f(x) = \begin{cases} \frac{1}{9} x e^{-x/3} & , \quad x > 0 \\ 0 & , \quad elsewhere. \end{cases}$$

The probability $P(0 < X < 3)$ is calculated as

$$P(0 < X < 3) = \int_{x=0}^{3} f(x)\,dx = \frac{1}{9} \int_{x=0}^{3} x e^{-x/3}\,dx = \left. \frac{-3(x+3)e^{-x/3}}{9} \right|_{x=0}^{3} = -\left. \frac{(x+3)e^{-x/3}}{3} \right|_{x=0}^{3} = 1 - 2e^{-1}.$$

Again, using the properties of the Gamma function, we can calculate any moment of the distribution as

$$E(X^k) = \int_{x=0}^{\infty} x^k f(x)\,dx = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{x=0}^{\infty} x^{k+\alpha-1} e^{-x/\beta} dx = \frac{\beta^{\alpha+k}\Gamma(\alpha+k)}{\beta^{\alpha}\Gamma(\alpha)} = \frac{\beta^k\,\Gamma(\alpha+k)}{\Gamma(\alpha)}.$$

For $k = 1$

$$E(X) = \frac{\beta\,\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta\,\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\,\beta$$

and for $k = 2$

$$E(X^2) = \frac{\beta^2\,\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\beta^2\,\alpha(\alpha+1)\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2$$

and therefore the variance of Gamma distribution is

$$Var(X) = E(X^2) - (E(X))^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2 .$$

The moment generating function of the Gamma distribution (for $t > 1/\beta$) is

$$M_X(t) = E(e^{tX}) = \int_{x=0}^{\infty} e^{tx} f(x)\,dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{x=0}^{\infty} e^{tx} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{x=0}^{\infty} x^{\alpha-1} e^{-x(1-\beta t)/\beta} dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{\beta}{1-t\beta} \right)^\alpha \Gamma(\alpha) = \left(\frac{1}{1-t\beta} \right)^\alpha = \frac{1}{(1-\beta t)^\alpha}.$$

As it is known, we can calculate the moments of random variable from the moment generating function as

$$E(X^k) = \left. \frac{d^k M_X(t)}{dt^k} \right|_{t=0} .$$

The moment generating function of the Gamma distribution is calculated as

$$M_X(t) = 1/(1-\beta t)^\alpha .$$

First two moments of the Gamma distribution can be obtained from the derivatives of the moment genetaring functio as

$$E(X) = \left. \frac{d\,M_X(t)}{dt} \right|_{t=0} = \left. \frac{\alpha\beta}{(1-\beta t)(1-\beta t)^\alpha} \right|_{t=0} = \alpha\beta$$

and

$$E(X^2) = \left. \frac{d^2 M_X(t)}{dt^2} \right|_{t=0} = \left. \left(\frac{\alpha^2\beta^2}{(1-\beta t)^2 (1-\beta t)^\alpha} + \frac{\alpha\beta^2}{(1-\beta t)^2 (1-\beta t)^\alpha} \right) \right|_{t=0} = \alpha^2\beta^2 + \alpha\beta^2 = \beta^2\alpha(\alpha+1).$$

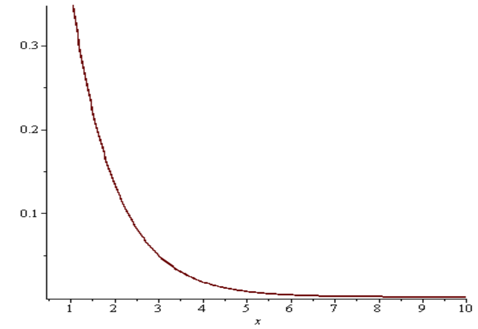
There are some well-known statistical distributions (like exponential, chi-square, Weibull, etc.) which are special case of the Gamma distribution.

2a. <u>Exponential Distribution</u>:

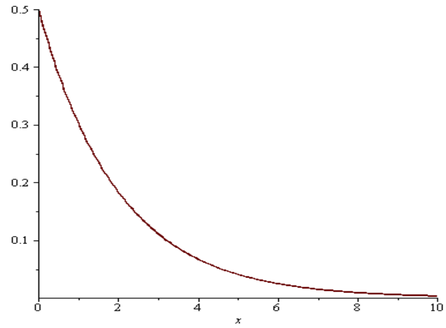
Let $X \sim Gamma(\alpha, \beta)$ and take $\alpha = 1$. Then we say that X is exponentially distributed with parameter β and denoted by $X \sim \exp(\beta)$. Then the probability density function of X will be

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & , \quad x > 0 \\ 0 & , \quad \textit{elsewhere.} \end{cases}$$

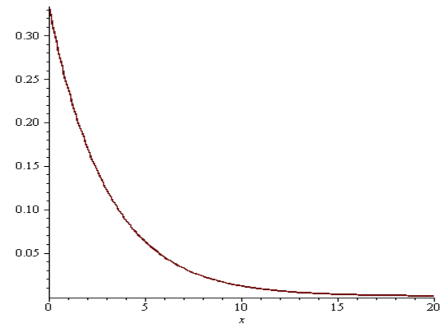
The graph of the probability density function of exponential distribution for different values of β are given below:



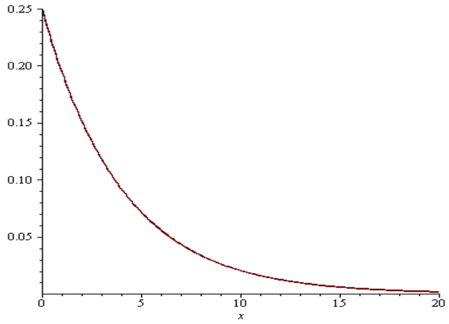
$\beta = 1$



$\beta = 2$



$\beta = 3$



$\beta = 4$

As it is seen from the graphs of the probability density function, the shape of the distribution depends heavily on the parameter β.

This distribution has many applications in the real life. Suppose you are producing an electronic item and you want to put a warranty on it. The item having longer warranty is expected to be sold more. If the life time of the product is less than the warranty, you may have sold more products but you will have to return much of the products sold. At the beginning, you may seem to make more money but if the warranty is more than expected life time you may lose more money. Therefore, you must correctly estimate the warranty period. The durability of such products is generally modeled by exponential distribution. Later, we are going to discuss such problems in the point estimation.

Since, the exponential distribution is a special case of the Gamma distribution, the Gamma distribution can be considered as a sum of independent exponential distribution.

Since, the exponential distribution is a special case of the Gamma distribution, the mean and the variance can be obtained by putting $\alpha = 1$. Simply, if $X \sim \exp(\beta)$, then

$E(X) = \beta$, $Var(X) = \beta^2$ and $M_X(t) = 1/(1-\beta t)$, $t > 1/\beta$.

Now, assume that $X \sim \exp(\beta)$ and s is any positive reel number. Then the probability that X being greater than s can be calculated as,

$$P(X > s) = \int_{x=s}^{\infty} f(x)\,dx = \frac{1}{\beta}\int_{x=s}^{\infty} e^{-x/\beta}dx = -e^{-x/\beta}\Big|_{x=s}^{\infty} = e^{-s/\beta}.$$

Now, let s and t be two positive real number such that $s > t$ and let us try to calculate the conditional probability of X being greater than s when $X > t$ is given. That is, we want to calculate $P(X > s \mid X > t)$. Using the probability obtained above ($P(X > s) = e^{-s/\beta}$) we can calculate the conditional probability as

$$P(X > s \mid X > t) = \frac{P(X > s, X > t)}{P(X > t)} = \frac{P(X > s)}{P(X > t)} = \frac{e^{-s/\beta}}{e^{-t/\beta}} = e^{-(s-t)/\beta} = P(X > s - t).$$

That is, we have the following equation

$$\boxed{P(X > s \mid X > t) = P(X > s - t).}$$

This equality is known to be the “memoryless” property of the random variable X. As it is mentioned before, there are so many continuous distributions in the literature. Among those continuous distribution is the one that satisfies the memoryless property.

2b. Chi-Square Distribution:

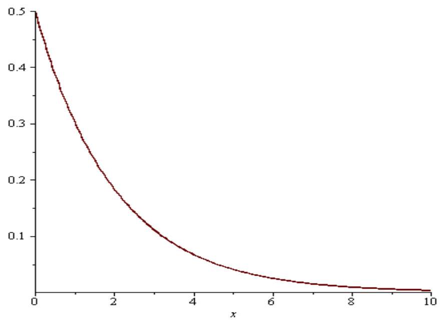
In statistics and any other branch of science, in order to make any inference about the unknowns of the populations we usually use four main distributions. Chi-square distribution is one of these four distributions. These distributions are also known as sample distributions. We are going to discuss these distributions later. Chi-square distribution can also be obtained from the normal distribution (we will discuss later) but here it can be obtained as a special case of Gamma distribution. Let $X \sim Gamma(\alpha, \beta)$ and take $\alpha = p/2$ and $\beta = 2$. Here, p is any positive integer ($p \in \mathbb{N}$). Then the probability density function of X will be

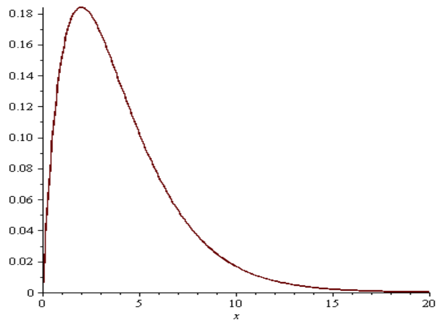
$$f(x) = \begin{cases} \dfrac{1}{\Gamma(p/2)\,2^{p/2}}\, x^{(p/2)-1} e^{-x/2} & , \quad x > 0 \\ 0 & , \quad elsewhere. \end{cases}$$

It is a probability density function because it is a special case of the Gamma distribution. If the random variable X has a probability density function as given above then we say that X is distributed as chi-square with p degrees of freedom and denoted by $X \sim \chi^2_p$. The mean and the variance can be calculated as

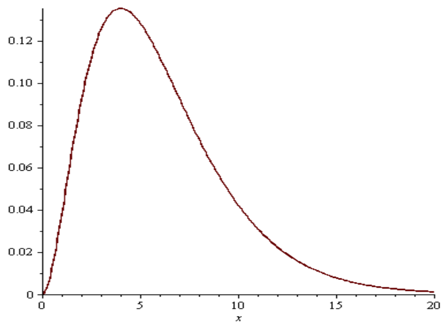
$$E(X) = \alpha\beta = (p/2)2 = p \text{ and } Var(X) = \alpha\beta^2 = (p/2)2^2 = 2p$$

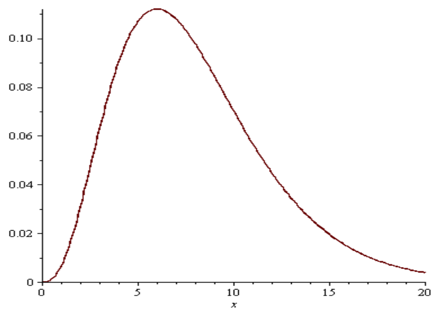
and the moment generating function (for $t < 1/2$) $M_X(t) = 1/(1-2t)^{p/2}$. The mean of chi-square distribution is the number of degrees of freedom and the variance is the two times number of degrees of freedom. The graphs of the probability density functions for $p=2$, $p=4$, $p=6$ and $p=8$ are given below.





$p=2$ $\qquad$ $p=4$





$p=6$ $\qquad$ $p=8$

Any moment of the chi-square distribution can be calculated by using the properties of the Gamma function as

$$E(X^k)=\Gamma\left(\frac{p}{2}+k\right)\left[\Gamma\left(\frac{p}{2}\right)\right]^{-1}2^k$$

because

$$E(X^k)=\int_{x=0}^{\infty} x^k f(x)\,dx=\frac{1}{\Gamma(p/2)2^{p/2}}\int_{x=0}^{\infty} x^k x^{(p/2)-1}e^{-x/2}\,dx$$

$$=\frac{1}{\Gamma(p/2)2^{p/2}}\int_{x=0}^{\infty} x^{k+(p/2)-1}e^{-x/2}\,dx=\frac{\Gamma\left(\frac{p}{2}+k\right)2^{k+(p/2)}}{\Gamma(p/2)2^{p/2}}=\frac{\Gamma\left(\frac{p}{2}+k\right)}{\Gamma\left(\frac{p}{2}\right)}2^k.$$

Again, the shape of the distribution is heavily depends on the degrees of freedom. This distribution has many application in statistics and any other branch of science. For example, the chi-square distribution is used to make any statistical inference about the population variance.

We are going to discuss some other applications in estimation and hypothesis testing problems. It is important to note that the sum of independent chi-square distribution is also distributed as chi-square. That is, is

$X_1 \sim \chi_p^2$ and $X_2 \sim \chi_q^2$ then $X_1 + X_2 \sim \chi_{p+q}^2$

when X_1 and X_2 are independent.

2b. **Weibull Distribution:**

Let $X \sim \exp(\beta)$ and γ be any positive reel number. The probability density function of $Y = X^{\gamma}$ is

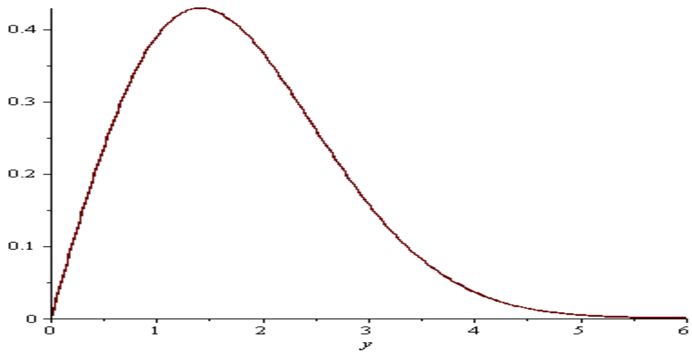
$$f(y) = \begin{cases} \dfrac{\gamma}{\beta} y^{\gamma-1} e^{-y^{\gamma}/\beta} & , \quad y > 0, \beta > 0, \gamma > 0 \\ 0 & , \quad elsewhere. \end{cases}$$

The mean and the variance of Weibull distribution is calculated as

$$E(Y) = \beta^{1/\gamma}\Gamma((\gamma+1)/\gamma), \qquad E(Y^2) = \beta^{2/\gamma}\Gamma((\gamma+2)/\gamma),$$

$$Var(Y) = \beta^{2\delta}[\Gamma(2\delta+1) - (\Gamma(2\delta+1))^2] \text{ , where } \delta = 1/\gamma .$$

The graph of probability density function of the Weibull distribution for $\gamma = 2$ and $\beta = 4$ is given below.



The probability density function of Weibull distribution, $\gamma = 2$ and $\beta = 4$

3. **Beta Distribution:**

Before we introduce the Beta distribution, let us give the Beta function first. The Beta function is defined as

$$Beta(\alpha,\beta) = \int_{x=0}^{1} x^{\alpha-1}(1-x)^{\beta-1}dx \ .$$

The function can also be written in terms of the Gamma function as

$$Beta(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and therefore we can define a probability density function of a random variable

$$f(x)=\begin{cases}\dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{Beta(\alpha,\beta)} & , \quad 0<x<1\\ 0 & , \quad elsewhere.\end{cases}$$

As it is seen, the function is obviously a probability density function. If a random variable X has a probability density function as given above, we say that X is distributed as Beta with parameters α and β, denoted by $X \sim Beta(\alpha,\beta)$. Any moment of the Beta distribution can be calculated by using the beta function defined above. The k .th moment is calculated as

$$E(X^k)=\frac{1}{Beta(\alpha,\beta)}\int_{x=0}^{1} x^k x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{x=0}^{1} x^{k+\alpha-1}(1-x)^{\beta-1}dx$$
$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}}\frac{\Gamma(\alpha+k)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+k+\beta)}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k+\beta)}.$$

Using this equality, first two moments of Beta distribution are

$$E(X)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\alpha\ \Gamma(\alpha)}{(\alpha+\beta)\Gamma(\alpha+\beta)}=\frac{\alpha}{(\alpha+\beta)}$$

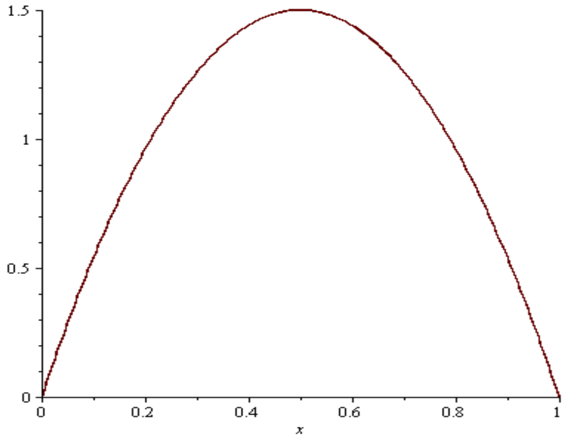
and

$$E(X^2)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+2+\beta)}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\alpha(\alpha+1)\ \Gamma(\alpha)}{(\alpha+\beta)(\alpha+\beta+1)\Gamma(\alpha+\beta)}$$
$$=\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}.$$

and therefore the variance of the beta distribution is

$$Var(X)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

The graph of the probability density function and some probabilities of the Beta distribution (for $\alpha=\beta=2$) is given below.



$$P(0 < X < 0.2) = \int_{x=0}^{0.2} f(x)dx = 0.104$$

$$P(0 < X < 0.4) = \int_{x=0}^{0.4} f(x)dx = 0.352$$

$$P(0 < X < 0.5) = \int_{x=0}^{0.5} f(x)dx = 0.500$$

$$P(0 < X < 0.8) = \int_{x=0}^{0.8} f(x)dx = 0.896$$

The graph of Beta(2,2) distribution and some probabilities

4. Cauchy Distribution:

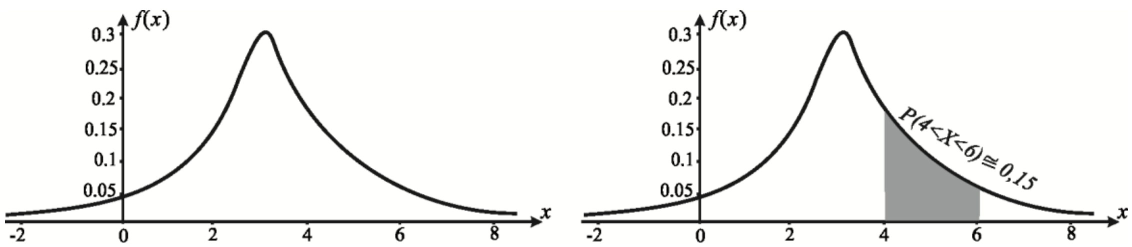
If the random variable X has the following probability density function, we say that X is distributed as Cauchy with parameter θ and denoted by $X \sim Cauchy(\theta)$:

$$f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}\ ,\ x \in \mathbb{R}\,.$$

This is aactually a probability density function because

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}dx = \frac{1}{\pi}\left(\arctan(\infty)-\arctan(-\infty)\right) = \frac{1}{\pi}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = 1.$$

The main property of the Cauchy distribution is that the distribution has no moment. That is no moment exists. The distribution is symmetric around θ and the graph of the distribution (for $\theta = 3$) is given below.



The graph of the probability density function of Cauchy distribution for $\theta = 3$.

Let $X \sim Cauchy(\theta = 3)$ then we calculate the following probabilities

$$P(4 < X < 6) = \int_4^6 f(x)\,dx = \frac{1}{\pi}\int_4^6 \frac{1}{1+(x-3)^2}\,dx = \frac{1}{\pi}\arctan(x-3)\Big|_{x=4}^{6}$$

$$= \frac{1}{\pi}\left(\arctan(3) - \arctan(1)\right) \cong \frac{1}{\pi}\left(\frac{2\pi}{5} - \frac{\pi}{4}\right) = \frac{3}{20} = 0.15$$

and

$$P(3 < X < 4) = \frac{1}{\pi}\int_4^6 \frac{1}{1+(x-3)^2}\,dx = \frac{1}{\pi}\arctan(x-3)\Big|_{x=3}^{4}$$

$$= \frac{1}{\pi}\left(\arctan(1) - \arctan(0)\right) = \frac{1}{\pi}\left(\frac{\pi}{4} - 0\right) = \frac{1}{4} = 0.25.$$

5. <u>Normal Distribution</u>:

There is no doubt that the normal distribution is the most used distribution in statistics in many scientific fields. One of the reasons is related to the central limit theorem, which we will examine later. According to the central limit theorem, almost all distributions in average converge to the normal distribution. Almost all statistical inference about the parameters depends on the assumption of normality of data. In cases where the normality is violated, we use some techniques (such as transformations) the normality assumption is tried to be provided. Based on the normally distributed data, we try to make some statistical inferences about the parameters. Regardless of what is written about the importance of normal distribution in practice, there will still be something missing.

If the random variable X has the following probability density function ($\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$),

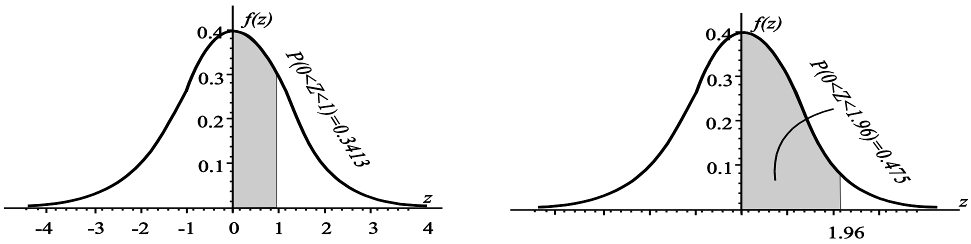
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad , x \in \mathbb{R}$$

we say that X is normally distributed with mean μ and the variance σ^2 and denoted by $X \sim N(\mu, \sigma^2)$.

If $X \sim N(\mu, \sigma^2)$ then the random variable $Z = (X - \mu)/\sigma$ is again normally distributed with mean 0 and variance 1. This distribution is known to be the standart normal distribution and denoted by $Z \sim N(0,1)$. The probability density function of the standart normal distribution is given by

$$f(z) = \frac{1}{\sqrt{2\pi}}\ e^{-z^2/2} \quad , z \in \mathbb{R}.$$

In many statistical textbook,ü the probabilities of standart normal distributions have been tabulated. For example, when $Z \sim N(0,1)$ we can find the probability $P(0 < Z < 1)$ from the standart normal table as 0.3413. This probability is actually is the area under the curve given below. The probability distribution is symmetric around the mean (here 0) and therefore, we can use this property to look at any other probabilities. For example, $P(-1 < Z < 0)$ and $P(0 < Z < 1)$ have the probability.



The probability density function of standart normal distribution and some probabilities

<u>Note:</u> A real valued function h is said to be an even function for all $x \in \mathbb{R}$, $h(-x) = h(x)$ and it is an odd function if $h(-x) = -h(x)$. On the other hand, a product of two even function is even, product of two odd functions is even. But the product of an even function and an off function is odd.

According to this definition, since

$$h(z) = e^{-z^2/2} = e^{-(-z)^2/2} = h(-z)$$

the function $h(z) = e^{-z^2/2}$ is even. On the other hand, the function $g(z) = z$ is odd (because $g(-z) = -z = -g(z)$). As you may seen in your calculas class, for any positive reel number a, we have

$$\int_{-a}^{a} f(z)\,dz = \begin{cases} 0 & , \quad f \text{ is an odd function} \\ 2\int_{0}^{a} f(z)\,dz & , \quad f \text{ is an even function.} \end{cases}$$

Note that the function $f(z) = z\,e^{-z^2/2}$ is odd and therefore the first moment of the standart normal distribution is

$$E(Z) = \int_{-\infty}^{\infty} z\, f(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\,e^{-z^2/2} dz = 0 .$$

The second moment of the distribution is calculated with the computer (Maple VIII) as

$$E(Z^2) = \int_{-\infty}^{\infty} z^2 \, f(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \, e^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 \, e^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2} = 1.$$

Thus the variance of the standart normal distribution is

$$Var(Z) = E(Z^2) - (E(Z))^2 = 1.$$

Notice that if $Z = (X - \mu)/\sigma$ then $X = \mu + \sigma Z$ the mean and variance of $X \sim N(\mu, \sigma^2)$ distribution are calculated as

$$E(X) = E(\mu + \sigma Z) = \mu + \sigma E(Z) = \mu \text{ and } Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2 .$$

In the above discussion, using the standart normal table we can find the probabilities. If the random variable X is normmaly distributed with mean μ and variance σ^2 ($X \sim N(\mu, \sigma^2)$), we can found the probabilities (for X) by transforming to standard normal distribution. For example, assume that $X \sim N(\mu = 100, \sigma^2 = 100)$ and let us try to find $P(100 < X < 110)$. Since, $\mu = 100$ and $\sigma = 10$ we have $Z = (X - 100)/10 \sim N(0,1)$ and therefore the probability $P(100 < X < 110)$ can be calculated as

$$P(100 < X < 110) = P\left(\frac{100-100}{10} < \frac{X-100}{10} < \frac{110-100}{10}\right) = P(0 < Z < 1) = 0.3413.$$

The probability density function of the standard normal random variable was given before,

$$f(z) = \frac{1}{\sqrt{2\,\pi}} \; e^{-z^2/2} \quad , z \in \mathbb{R}..$$

Now let us try to show that it is actually a probability density function. Since the function $f(z)$ is symmetric around zero, we have,

$$\int_{-\infty}^{\infty} f(z)\,dz = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = \frac{2}{\sqrt{2\,\pi}} \int_{0}^{\infty} e^{-z^2/2} dz = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$$

On the other hand, we can not integrate the funtion

$$\int e^{-z^2/2} dz$$

is not integrable. However, we can integrate

$$\int_{0}^{\infty} e^{-z^2/2} dz$$

by using the polar coordinate techniques. Notice that we can write the following identities:

$$1=\int_{-\infty}^{\infty} f(z)\,dz \Leftrightarrow \int_{0}^{\infty} e^{-z^2/2}dz=\sqrt{\frac{\pi}{2}} \Leftrightarrow \left(\int_{0}^{\infty} e^{-z^2/2}dz\right)^2=\frac{\pi}{2}$$

$$\Leftrightarrow \left(\int_{0}^{\infty} e^{-z^2/2}dz\right)\left(\int_{0}^{\infty} e^{-u^2/2}du\right)=\frac{\pi}{2} \Leftrightarrow \int_{0}^{\infty}\int_{0}^{\infty} e^{-(z^2+u^2)/2}\,dz=\frac{\pi}{2}.$$

Therefore, in order to show that

$$\int_{-\infty}^{\infty} f(z)\,dz=1$$

it is enough to show

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-(z^2+u^2)/2}\,dzdu=\frac{\pi}{2}.$$

Consider the change of variables as for $r>0$ and $0<\theta<\pi/2$

$z=r\cos(\theta)$ ve $u=r\sin(\theta)$

then it is obvious that (from jacobiens) $dz\,du=r\,dr\,d\theta$ and therefore the double integral turns out to be

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-(z^2+u^2)/2}\,dz\,du=\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2/2}r\,dr\,d\theta$$

$$=\int_{r=0}^{\infty} r\,e^{-r^2/2}\,dr\int_{\theta=0}^{\pi/2} d\theta=\frac{\pi}{2}\int_{r=0}^{\infty} r\,e^{-r^2/2}\,dr=\frac{\pi}{2}.$$

Thus the function $f(z)$ is a probability density function.

The moment generating function of the standard normal random variable is found to be $M_Z(t)=e^{t^2/2}$.

The moment generating function of $X \sim N(\mu,\sigma^2)$ distribution can be obtained from the equality $X=\mu+\sigma Z$ as

$$M_X(t)=M_{\mu+\sigma Z}(t)=E(e^{(\mu+\sigma Z)t})=e^{\mu t}M_X(\sigma t)=e^{\mu t+\sigma^2 t^2/2}.$$

Example a) Let $X \sim N(\mu=100,\sigma^2=100)$. Some of the probabilities have been calculated and the areas under the curve are given the the following.

$$P(90<X<110)=P\left(\frac{90-100}{10}<\frac{X-100}{10}<\frac{110-100}{10}\right)$$

$$=P(-1<Z<1)=2P(0<Z<1)=2(0.3413)=0.6826$$

This probability is the shaded area under the curve given in (a). The probability $P(110 < X < 120)$ is calculated as

$$P(110 < X < 120) = P\left(\frac{110-100}{10} < \frac{X-100}{10} < \frac{120-100}{10}\right)$$
$$= P(1 < Z < 2) = P(0 < Z < 2) - P(0 < Z < 1) = 0.4772 - 0.3413 = 0.1359$$

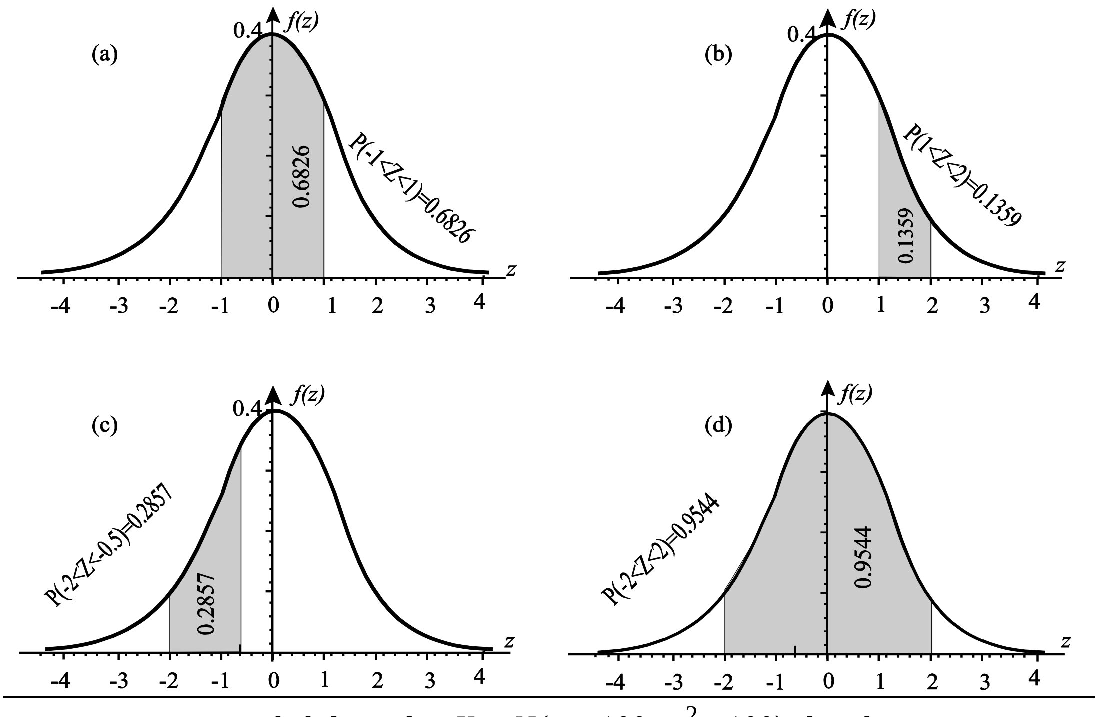
and this probability is also the shaded area under the curve given in (b). Moreover, the probability $P(80 < X < 95)$ is

$$P(80 < X < 95) = P\left(\frac{80-100}{10} < \frac{X-100}{10} < \frac{95-100}{10}\right) = P(-2 < Z < -0.5)$$
$$= P(0.5 < Z < 2) = P(0 < Z < 2) - P(0 < Z < 0.5) = 0.4772 - 0.1915 = 0.2857.$$

Again, the probability is the shaded area given in © below. Finally, let us calculate the probability $P(|X-100| \leq 20)$. This can be calculated as

$$P(|X-100| \leq 20) = P(-20 \leq X-100 \leq 20) = P(-2 \leq Z \leq 2)$$
$$= 2P(0 \leq Z \leq 2) = 2(0.4772) = 0.9544$$

and the area is shaded as in the figure (c) below.

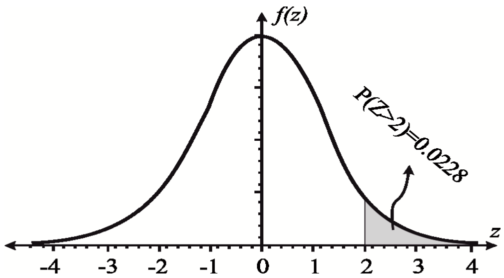


Some probabilities for $X \sim N(\mu = 100, \sigma^2 = 100)$ distribution

b) Assume that the test scores for a statistic class is normally distributed with mean 70 and the variance 100. Four students get more than 90 from the exam find the number of total students in the examination. In order to find the total number of students it will be enough to calculate the probability that students who get more than 90 in the exam. That is, we need to calculate $P(X > 90)$. This probability is easily calculated as

$$P(X > 90) = P\left(\frac{X-70}{10} > \frac{90-70}{10}\right) = P(Z > 2) = 0.0228$$

which means that approximately 2.5% of all students get more than 90 in the exam.



If the 2.5% of all students is 4, then to total number of students will be $400/(2.5) = 160$. That is, 160 students have been in the examination. The probability calculated above is the shaded are under the normal curve given above.

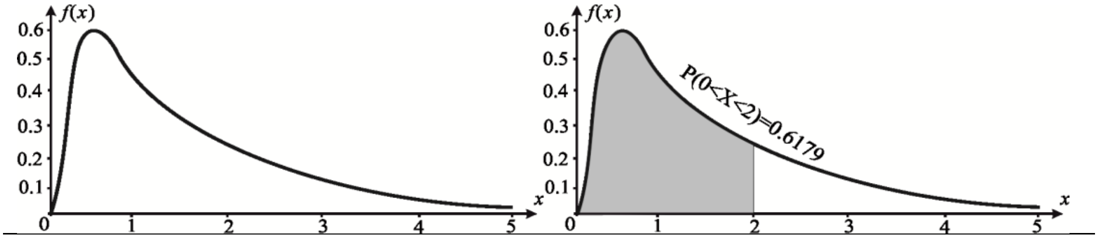
Finally notice that sum of independent normally distributed random vari.ables is also normally distributed. That is, if $X_1, X_2, ..., X_k$ independent and $N(\mu_i, \sigma_i^2)$ distributed random variables, then $X = X_1 + X_2 + ... + X_k \sim N(\mu, \sigma^2)$ where $\mu = \mu_1 + ... + \mu_k$ and $\sigma^2 = \sigma_i^2 + ... + \sigma_k^2$.

6. <u>Log-Normal Dağılım</u>:

We say that the random variable X is distributed as log-normal if it has a probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} e^{-\frac{1}{2\sigma^2}(\log(x)-\mu)^2} \quad , x \in \mathbb{R}^+, \ \mu \in \mathbb{R}, \ \sigma > 0$$

and denoted by $X \sim \log N(\mu, \sigma^2)$. If $X \sim \log N(\mu, \sigma^2)$ then $Y = \log(X) \sim N(\mu, \sigma^2)$. The distribution has some important applications especially in economics. As it is mentioned above, the normality is one of the main assumption in order to make any statistical inference about the unknowns of the population. Most of the economic data have been modeled as log normal. In order to achieve the normality before the analysis, we use the logaritmic transformation before the analysis and based on the transformed data we do the inferences. The graph of the probability density function of the distribution (for $\mu = 0$ and $\sigma = 1$) is given below.



Probablity density function of the log normal distribution ($\mu = 0$ and $\sigma = 1$)

If $X \sim \log N(0,1)$ then let us try to calculate $P(0 < X < 1)$. The probabilityh can be calculated directly as

$$P(0 < X < 1) = \int_0^1 f(x)dx = \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{x} e^{-(\log(x))^2/2} dx \ , \ \log(x) = u \Rightarrow \frac{1}{x} dx = du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-u^2/2} du = P(-\infty < Z < 0) = 0.5.$$

and similarly, the $P(0 < X < 2)$ probability is calculated as

$$P(0 < X < 2) = \int_0^2 f(x)dx = \frac{1}{\sqrt{2\pi}} \int_0^2 \frac{1}{x} e^{-(\log(x))^2/2} dx \ , \ \log(x) = u \Rightarrow \frac{1}{x} dx = du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\log(2)} e^{-u^2/2} du = P(-\infty < Z < \log(2)) \cong P(-\infty < Z < 0.3) = 0.6179.$$

This probability is the shaded area under the curve given above. The mean and the variance of the distribution are

$$E(X) = e^{\mu + \sigma^2/2} \quad \text{and} \quad Var(X) = e^{2(\mu + \sigma^2)} - e^{2\mu + \sigma^2}.$$