

WEEK 12

10. Hypothesis Testing and Confidence Intervals

In the previous sections, we have discussed the basic probability concept and estimation techniques about the population parameters. After deciding a better estimator, based on the observed values we give an estimate. The basic question is to decide how this estimate represents the population. We know that the normality is very important in statistic and any other field of science. If we do not have the normality, we transform the data to achieve the normality. When we have large number of observations, we use the central limit theorem to achieve the normality assumption. Therefore, in the section we are going to look at some statistical inferences about the normal parameters. That is we will assume that a random sample X_1, X_2, \dots, X_n comes from a $N(\mu, \sigma^2)$ population.

Definition (Hypothesis) A hypothesis is any claim about the population parameter.

For example someone may claim that the average of a statistic class is less than 75. This is a claim about the population mean. If the mean of the population is μ then the hypothesis will be $\mu < 75$. We talk about two hypotheses. One is known as the null hypothesis and denoted by H_0 and the other is the alternative hypothesis and denoted by either H_a or H_1 and we try to test H_0 against the H_a . For example, if we want to test the null hypothesis $H_0 : \mu \leq 75$ based on the observed values x_1, x_2, \dots, x_n the alternative will be $H_a : \mu > 75$.

Suppose that someone in a class claim that the mean of STAT class is too low (say less than or equal to 75). In order to test his/her claim we perform an experiment (give an examination) and collect the data x_1, x_2, \dots, x_n . According to the data values, we may reject the null hypothesis or we may not reject the null (fail to reject null or accept). That is there are two possibilities. Based on the observed values

- we reject the null
- we fail to reject the null

Moreover, in the real life there are two possibilities:

- The null is true (that is H_0 is true, or $\mu \leq 75$)
- the alternative is true (that is H_a is true or $\mu > 75$).

Let the true mean be less than or equal to 75 (that is $\mu \leq 75$). Then based on the sample values we may reject the null hypothesis. In this case we made an error. That is, when H_0 is true based on the sample it is possible that we may reject the null hypothesis, this is *the type I error*. Similarly, when the null hypothesis is false (or the alternative is true), it is possible that we may fail to reject the null hypothesis. In this case we make another error. This is called the *type II error*.

- **Type I error:** Reject the null (H_0) when it is true.
- **Type II error:** Fail to reject the null (H_0) when the alternative is true (or H_0 is false).

In summary, the possibilities have been given in the following table.

True Hypothesis	Decision	
	H_0	H_a
	H_0	Correct Decision
H_a	Type II Error	Correct Decision

The probability of the type I error is called the significance level of the test and usually denoted by α . That is,

$$P(\text{Type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true}) = \alpha.$$

Similarly,

$$P(\text{Type II error}) = P(\text{fail to reject } H_0 \mid H_0 \text{ is false}) = \beta$$

and $1 - P(\text{Type II error})$ is the power of the test.

In general, the power of the test is the probability of rejecting the null hypothesis which is a function of the sample size and μ .

Hypothesis Testing for Normal Mean:

In this section we are going to investigate the following three hypothesis testing problems. The test statistics and the critical values changes according to the case whether the variance is known or not. Therefore firstly we are going to investigate the case that the variance of the normal population σ^2 is known. The significance level α and μ_0 will be given and based on the observed values x_1, x_2, \dots, x_n we will test the following hypotheses.

CASE I: Hypothesis Testing for the normal population mean when σ^2 is known.

Let X_1, X_2, \dots, X_n be a random sample from $N(\mu, \sigma^2)$ distribution and assume that σ^2 is known. Then consider the following hypotheses. Remember that if the data is collected from a non-normal population, we use the central limit theorem to achieve the normality.

- a) $H_0 : \mu = \mu_0$ against $H_a : \mu > \mu_0$ or equivalently $H_0 : \mu \leq \mu_0$ against $H_a : \mu > \mu_0$
- b) $H_0 : \mu = \mu_0$ against $H_a : \mu < \mu_0$ or equivalently $H_0 : \mu \geq \mu_0$ against $H_a : \mu < \mu_0$
- c) $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$

The significance level α and μ_0 will be given and based on the observed values x_1, x_2, \dots, x_n we will test the following hypotheses.

- a) Consider the first hypothesis testing problem $H_0 : \mu \leq \mu_0$ against $H_a : \mu > \mu_0$.

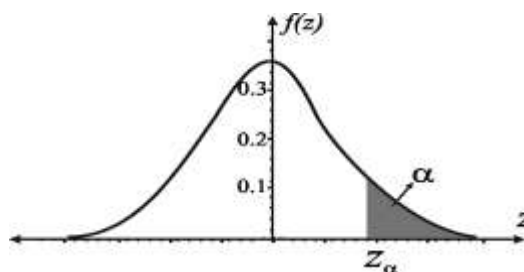
Let X_1, X_2, \dots, X_n be a random sample from $N(\mu, \sigma^2)$ distribution and assume that σ^2 is known. Remember that

$$\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma} \sim N(0,1).$$

We reject the null hypothesis ($H_0 : \mu \leq \mu_0$ or equivalently $H_0 : \mu = \mu_0$) against the alternative $H_a : \mu > \mu_0$ at the level α if $z_h > z_\alpha$ where

$$z_h = \frac{\sqrt{n}(\bar{x}_n - \mu_0)}{\sigma} \text{ and } P(Z > z_\alpha) = \alpha.$$

The value of z_α can be found from a standard normal table. That is if the calculated value z_h is greater than the critical value (z_α) we reject the null hypothesis. Therefore the rejection region of the test can be written as $\mathcal{R} = \{x : z_h > z_\alpha\}$ which is the shaded area in the following figure.



The Rejection Region of the hypothesis testing of $H_0 : \mu = \mu_0$ against $H_a : \mu > \mu_0$

Example: Assume that the test scores for BAS are normally distributed with mean μ and variance $\sigma^2 = 64$. Students claim that the instructor's grading is very poor and therefore the

average is too low (say less than 70). Based on the 25 student's test scores given below, the student's claim at %5 level.

66	71	67	69	75	66	64	70	62	87	70	79	74
84	76	63	74	79	93	76	68	88	72	70	77	
25 Student's test scores												

We need to test the following hypothesis at 5% level. Since $\alpha = 0.05$ the critical value can be found from the standard normal table ($P(Z > z_\alpha) = \alpha$) as $z_\alpha = 1.645$ and we are going to reject the null hypothesis $H_0 : \mu \leq 70$ against the alternative $H_a : \mu > 70$ if $z_h > 1.645$. The hypotheses are

$$H_0 : \mu \leq 70 \text{ versus } H_a : \mu > 70 .$$

The mean and the value of the test statistic are calculated from the sample as

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i = 73.60 \quad \text{and} \quad z_h = \frac{\sqrt{n}(\bar{x}_n - \mu_0)}{\sigma} = \frac{\sqrt{25}(73.6 - 70)}{8} = \frac{5(3.6)}{8} = 2.25 .$$

Since, $z_h = 2.25 > 1.645 = z_\alpha$, we reject the null hypothesis at 5% level.

Notice that the estimated variance from the data is actually $s_n^2 = 64.75$ and since the variance is given we do not need to calculate the variance in this problem.

Now let us calculate the power of the test. As we know the power is the probability of rejecting the null hypothesis which is actually a function of μ and the sample size n . The power also depends on the significance level α . That is the power function

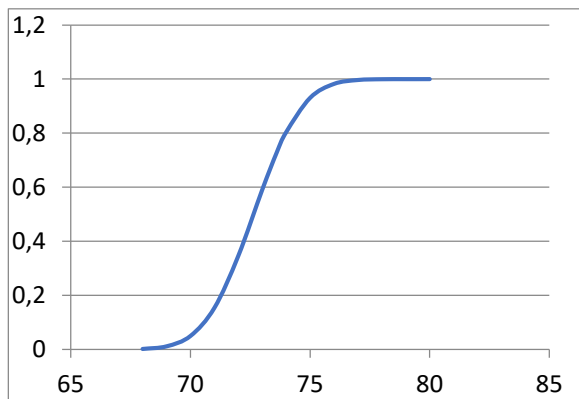
$$\begin{aligned} \beta(\mu) &= P_\mu(\text{reject } H_0) = P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma} > z_\alpha\right) = P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu + \mu - \mu_0)}{\sigma} > z_\alpha\right) \\ &= P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > z_\alpha - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}\right) = P\left(Z > z_\alpha - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}\right) \\ &= P\left(Z > 1.645 - \frac{\sqrt{25}(\mu - 70)}{8}\right) = P(Z > z_\alpha(\mu)). \end{aligned}$$

Here $z_\alpha(\mu) = z_\alpha - \sqrt{n}(\mu - \mu_0) / \sigma$. By using the normal table the power of the test for some values of μ are calculated below. As it is seen the power is an increasing function of μ . In our case for 5% level $z_\alpha = 1.645$ and $\mu_0 = 70$ and therefore $z_\alpha(\mu) = 1.645 - \sqrt{25}(\mu - 70) / 8$.

μ	68	69	70	71	72	73	73.6	74	75	76
$z_\alpha(\mu)$	2.895	2.270	1.645	1.020	0.395	-	-	-	-	-
			5			0.230	0.605	0.855	1.480	2.105

$\beta(\mu)$	0.001	0.011	0.05	0.153	0.346	0.590	0.727	0.803	0.930	0.982
	9	6		9	4	9	4	7	6	4

The graph of the power function is given in the following figure.



As it is seen the power increases when the μ increases. This is not always the case. It depends on the direction of the alternative hypothesis. However, the power function is always an increasing function of the sample size. The power at the calculated mean ($\bar{x}_n = 73.6$) is called the empirical power of the test.

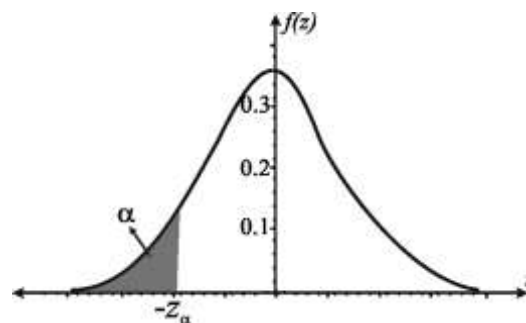
The power function of the test

b) Consider the first hypothesis testing problem $H_0 : \mu \geq \mu_0$ against $H_a : \mu < \mu_0$.

Let X_1, X_2, \dots, X_n be a random sample from $N(\mu, \sigma^2)$ distribution and assume that σ^2 is known. We reject the null hypothesis ($H_0 : \mu \geq \mu_0$ or equivalently $H_0 : \mu = \mu_0$) against the alternative $H_a : \mu < \mu_0$ at the level α if $z_h < -z_\alpha$ where

$$z_h = \frac{\sqrt{n}(\bar{x}_n - \mu_0)}{\sigma} \quad \text{and} \quad P(Z > z_\alpha) = \alpha.$$

The value of z_α can be found from a standard normal table. That is if the calculated value z_h is smaller than the critical value ($-z_\alpha$) we reject the null hypothesis. Therefore the rejection region of the test can be written as $\mathcal{R} = \{x : z_h < -z_\alpha\}$ which is the shaded area in the following figure.



The Rejection Region of the hypothesis testing of $H_0 : \mu = \mu_0$ against $H_a : \mu < \mu_0$

Example: Daily meat consumption (in kg) of a restaurant at the Kızılay square in Ankara is normally distributed with mean μ and variance $\sigma^2 = 36$. The owner of the restaurant wants to test whether the mean meat consumption is greater than or equal to 65 kg at 5% level against the alternative hypothesis that daily meat consumption is less than 65 kg based on 36-days consumption. The data is given below.

57	55	66	73	73	58	63	64	68	71	58	49	67	72	59	61	59	57
63	60	62	67	78	60	76	73	65	62	55	67	59	66	64	62	59	58

From the observations we have $\bar{x}_n = 63.5$ (and $s_n^2 = 42.4857$ but we do not need to calculate the variance because it is given). Here we want to test the hypothesis $H_0 : \mu \geq 65$ against the alternative of $H_a : \mu < 65$. From the standard normal table, $P(Z > z_\alpha) = 0.05$ we find the critical value $z_\alpha = 1.645$ and if $z_h < -z_\alpha$ we will reject the null hypothesis. Since,

$$z_h = \sqrt{n}(\bar{x}_n - 65) / \sigma = \sqrt{36}(63.5 - 65) / 6 = -1.5 > -1.645 = z_\alpha$$

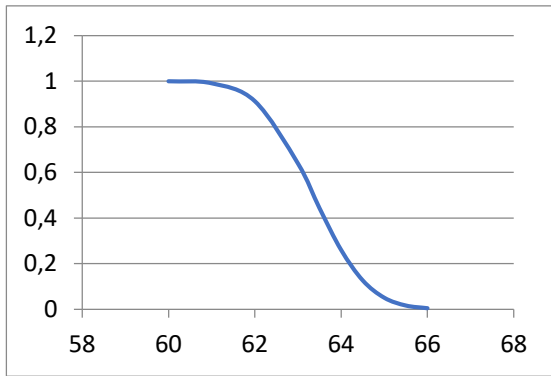
we fail to reject the null hypothesis. That is the value of the test statistic is in the rejection region and therefore we can conclude that the mean daily meat consumption can not be smaller than 65 kg. Now, let us calculate the power of the test. First the power function is

$$\begin{aligned} \beta(\mu) &= P_\mu(\text{reject } H_0) = P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma} < -z_\alpha\right) = P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu + \mu - \mu_0)}{\sigma} < -z_\alpha\right) \\ &= P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > < -z_\alpha - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}\right) = P\left(Z < -z_\alpha - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}\right) \\ &= P\left(Z < -1.645 - \frac{\sqrt{36}(\mu - 65)}{6}\right) = P(Z < -1.645 - (\mu - 65)) = P(Z < z_\alpha(\mu)). \end{aligned}$$

By using the standard normal table we calculate some of these probabilities (the powers) and we give these powers in the following table.

μ	60	61	62	63	63.5	64	64.5	65	65.5	66
$z_\alpha(\mu)$	3.355	2.355	1.355	0.355	-	-	-	-	-	-
					0.145	0.645	1.145	1.645	2.145	2.645
$\beta(\mu)$	0.999 6	0.990 7	0.912 3	0.638 7	0.442 4	0.259 5	0.126 1	0.05	0.015 9	0.004 1

The graph of the power function is given in the following figure.



As it is seen from the table, the power decreases when the actual mean increase. Moreover the empirical power (power at the calculated mean $\bar{x}_n = 63.5$) is 0.4424. That is, the power function is an increasing function of μ and graph of powers are given below.

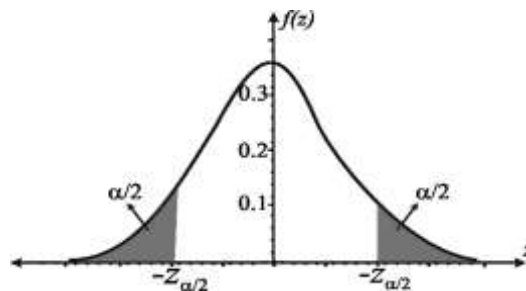
The power function of the test

c) Consider the first hypothesis testing problem $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$.

Let X_1, X_2, \dots, X_n be a random sample from $N(\mu, \sigma^2)$ distribution and assume that σ^2 is known. We reject the null hypothesis ($H_0 : \mu = \mu_0$) against the alternative $H_a : \mu \neq \mu_0$ at the level α if $|z_h| > z_{\alpha/2}$ where

$$z_h = \frac{\sqrt{n}(\bar{x}_n - \mu_0)}{\sigma} \text{ and } P(Z > z_\alpha) = \alpha.$$

The value of z_α can be found from a standard normal table. That is if the calculated value z_h in an absolute value is greater than the critical value ($z_{\alpha/2}$) we reject the null hypothesis. Therefore the rejection region of the test can be written as $\mathcal{R} = \{x : |z_h| > z_{\alpha/2}\}$ which is the shaded area in the following figure.



The Rejection Region of the hypothesis testing of $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$

Example: Assume that the lifetime of refrigerator is normally distributed with mean μ and variance $\sigma^2 = 2.25$. The producer wants to know the mean lifetimes of refrigerators in order to put a warranty on it. Based on randomly selected 25 refrigerator's lifetimes are given below table.

8.4	11.8	8.7	12.5	10.1	12.1	9.6	8.7	12.4	10.8	12.3	8.1	12.8
11.0	10.8	9.6	9.4	9.2	12.6	10.5	8.4	10.4	11.5	9.6	11.2	

Based on these observed values test whether the mean lifetime of the refrigerators is 10 years or not at 5% level.

We want to test $H_0 : \mu = 10$ against the alternative $H_a : \mu \neq 10$. From the observed values we have calculated $\bar{x}_n = 10.5$ and $s_n^2 = 2.22$ (in this question we do not need to calculate the sample variance because it is given) and from the standard normal table we find the critical value from $P(Z > z_{\alpha/2}) = 0.025$ as $z_{\alpha/2} = 1.96$ and the value of the test statistic

$$z_h = \sqrt{n}(\bar{x}_n - \mu_0) / \sigma = \sqrt{25}(10.5 - 10) / (1.5) \cong 1.667$$

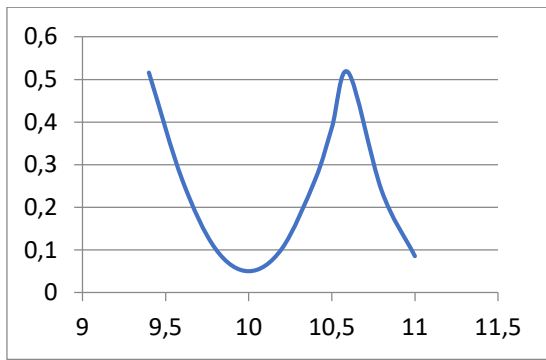
and since $|z_h| = 1.667 < 1.96 = z_{\alpha/2}$ we fail to reject the null hypothesis and at the 5% level we can conclude that the mean lifetime of the refrigerators is 10.5 years. What about the power of the test? The power function of this hypothesis testing problem is found as

$$\begin{aligned} \beta(\mu) &= P_{\mu}(\text{reject } H_0) = P_{\mu}\left(\left|\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma}\right| > z_{\alpha/2}\right) = 1 - P_{\mu}\left(\left|\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma}\right| \leq z_{\alpha/2}\right) \\ &= 1 - P_{\mu}\left(-z_{\alpha/2} \leq \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma} \leq z_{\alpha/2}\right) = 1 - P_{\mu}\left(-z_{\alpha/2} \leq \frac{\sqrt{n}(\bar{X}_n - \mu + \mu - \mu_0)}{\sigma} \leq z_{\alpha/2}\right) \\ &= 1 - P_{\mu}\left(-z_{\alpha/2} \leq \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \leq z_{\alpha/2}\right) \\ &= 1 - P_{\mu}\left(-z_{\alpha/2} - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \leq \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq z_{\alpha/2} + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}\right) \\ &= 1 - P_{\mu}\left(-z_{\alpha/2} - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \leq Z \leq z_{\alpha/2} + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}\right) = 1 - P(z_1(\mu) \leq Z \leq z_2(\mu)). \end{aligned}$$

In this problem $z_1(\mu) = -1.96 - 5(\mu - 10) / (1.5)$ and $z_2(\mu) = 1.96 - 5(\mu - 10) / (1.5)$. For some actual values the powers have been calculated in the below table. In order to calculate the powers all we need to do is to look at the standard normal table.

μ	9.4	9.6	9.8	10.0	10.2	10.4	10.5	10.6	10.8	11.0
$z_1(\mu)$	0.04	-0.63	-1.29	-1.96	-2.63	-3.29	-3.63	-3.96	-4.63	-5.29
$z_2(\mu)$	3.96	3.29	2.63	1.96	1.29	0.63	0.29	-0.04	-0.71	-1.37
$\beta(\mu)$	0.5159	0.2649	0.1028	0.05	0.1028	0.2649	0.3861	0.5159	0.2389	0.0853
$\beta(\mu) = 1 - P(z_1(\mu) \leq Z \leq z_2(\mu))$										

The graph of the power function is given in the following figure.



As it is seen from the table, the power decreases until $\mu = 10$ at the point the power is the significance level $\alpha = 0.05$ and after this point the power starts to increase. Moreover the empirical power (power at the calculated mean $\bar{x}_n = 10.5$) is 0.3861 which is an expecting situation because we failed to reject the null hypothesis.

The power function of the test

CASE II: Hypothesis Testing for the normal population mean when σ^2 is unknown

In the part, we are going to investigate the same hypotheses which are given in the previous case. That is we have the following hypothesis testing problem in hands

- a) $H_0 : \mu = \mu_0$ against $H_a : \mu > \mu_0$ or equivalently $H_0 : \mu \leq \mu_0$ against $H_a : \mu > \mu_0$
- b) $H_0 : \mu = \mu_0$ against $H_a : \mu < \mu_0$ or equivalently $H_0 : \mu \geq \mu_0$ against $H_a : \mu < \mu_0$
- c) $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$

As it is mentioned before, if the variance (σ^2) of the population is a parameter we use its estimator. Remember that when we have a random sample from a normal population we have

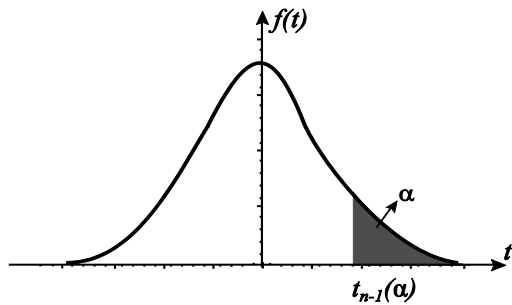
$$\sqrt{n}(\bar{X}_n - \mu) / S_n \sim t_{n-1}.$$

Therefore, we use the the value of the t statistics to make any statistical inference about the population mean when the variance is unknown. Similar to the normal distribution the probabilities for various degrees of freedoms have been tabulated for t distribution and these probabilities and critical values can be found in any statistics textbooks. In the following the rejection rules and rejection regions will be summarized for the above hypothesis testing problem and we will use the same example given in the previous case (known variance situation). The only difference is that we will assume that the variances of the populations will be unknown and therefore we will use the estimated variances. For all the hypotheses given above we use the same value of the test statistics

$$t_h = \frac{\sqrt{n}(\bar{x}_n - \mu_0)}{s_n}.$$

- a) $H_0 : \mu = \mu_0$ against $H_a : \mu > \mu_0$ or equivalently $H_0 : \mu \leq \mu_0$ against $H_a : \mu > \mu_0$.

For this hypothesis testing problem, we reject the null hypothesis at the significance level α if $t_h > t_{n-1}(\alpha)$ and the rejection region of the hypothesis is given in the following figure.



We reject the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_a : \mu > \mu_0$ if $t_h > t_{n-1}(\alpha)$ and the value of $t_{n-1}(\alpha)$ can be found in t table by using the significance level and the number of degrees of freedom.

The Rejection Region of the hypothesis testing of $H_0 : \mu = \mu_0$ against $H_a : \mu > \mu_0$

Example: Consider the previous example given in the case (a) above. Assume that the test scores for BAS are normally distributed with mean μ and variance σ^2 . Students claim that the instructor's grading is very poor and therefore the average is too low (say less than 70). Based on the 25 student's test scores given below, these student's claim at %5 level.

66	71	67	69	75	66	64	70	62	87	70	79	74
84	76	63	74	79	93	76	68	88	72	70	77	
25 Student's test scores												

We need to test the following hypothesis at 5% level. Since $\alpha = 0.05$ the critical value can be found from the t ($P(t_{24} > t_{24}(\alpha)) = \alpha$) as $t_{24}(\alpha) z_\alpha = 1.7109$ and we are going to reject the null hypothesis $H_0 : \mu \leq 70$ against the alternative $H_a : \mu > 70$ if $z_h > 1.7109$. The hypotheses are

$$H_0 : \mu \leq 70 \text{ versus } H_a : \mu > 70 .$$

The mean and the value of the test statistic are calculated from the sample as

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i = 73.60 \quad \text{and} \quad t_h = \frac{\sqrt{n}(\bar{x}_n - \mu_0)}{s_n} = \frac{\sqrt{25}(73.6 - 70)}{\sqrt{64.75}} = \frac{5(3.6)}{8.047} = 2.237 .$$

Since, $t_h = 2.237 > 1.7109 = t_{24}(0.05)$, we reject the null hypothesis at 5% level.

Notice that the standard deviation is very close to the actual standard deviation in the case that the variance is known. In the previous example we assume that the variance is known (it was 64) and the estimated variance (it is 64.75) are very close to each other. What about the

powers. Notice that the ampirical power was calculated (in the case that the variance is known) as **0.7227**. Now let us calculate the ampirical power of the test. The power function is calculated in a similar way and it is given by

$$\beta(\mu) = P\left(t_{24} > 1.7109 - 5(\mu - 70) / \sqrt{64.75}\right)$$

and the ampirical power (at the calculated mean $\bar{x}_n = 73.6$) is

$$\beta(73.6) = P\left(t_{24} > 1.7109 - 5(73.6 - 70) / \sqrt{64.75}\right) = P(t_{24} > -0.526) = 0.6982.$$

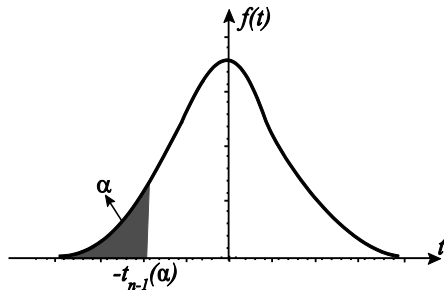
As it is seen the powers are different even the actual variance being very close to the calculated variance.

b) $H_0 : \mu = \mu_0$ against $H_a : \mu < \mu_0$ or equivalently $H_0 : \mu \geq \mu_0$ against $H_a : \mu < \mu_0$.

In order to test this hypothesis we calculate the value of the same test statistic as in the previous case ($t_h = \sqrt{n}(\bar{x}_n - \mu_0) / s_n$) and compare the critical value. As a rule, we reject the null hypothesis

$H_0 : \mu = \mu_0$ against the alternative $H_a : \mu < \mu_0$ if $t_h < -t_{n-1}(\alpha)$ where $P(t_{n-1} < -t_{n-1}(\alpha)) = \alpha$

. The rejection region of the test is given in the following figure.



We reject the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_a : \mu < \mu_0$ if $t_h < -t_{n-1}(\alpha)$ and the value of $t_{n-1}(\alpha)$ can be found in t table by using the significance level and the number of degrees of freedom.

The Rejection Region of the hypothesis testing of $H_0 : \mu = \mu_0$ against $H_a : \mu < \mu_0$

Example: Consider the meat consumption data given in the previous case. We will test the same hypothesis without assuming that the variance is known. That is, the variance is a parameter now. Daily meat consumption (in kg) of a restaurant at the Kızılay square in Ankara is normally distributed with mean μ and variance σ^2 . The owner of the restaurant wants to test wheather the mean meat consumption is greater than or equal to 65 kg at 5% level against the alternative hypothesis that daily meat consumption is less than 65 kg based on 36-days consumption. The data is given below.

57	55	66	73	73	58	63	64	68	71	58	49	67	72	59	61	59	57
63	60	62	67	78	60	76	73	65	62	55	67	59	66	64	62	59	58

From the observations we have $\bar{x}_n = 63.5$ and $s_n^2 = 42.4857$ and we want to test the hypothesis

$$H_0 : \mu \geq 65 \text{ against the alternative of } H_a : \mu < 65.$$

From the t table, $P(t_{35} < -t_{35}(\alpha)) = 0.05$ we find the critical value $t_{35}(0.05) = 1.6896$ and if $t_h < -t_{35}(0.05)$ we are going to reject the null hypothesis. Since,

$$t_h = \sqrt{n}(\bar{x}_n - 65) / s_n = \sqrt{36}(63.5 - 65) / \sqrt{42.4857} = -1.3808 > -1.6896 = -t_{35}(0.05)$$

we fail to reject the null hypothesis. That is the value of the test statistic is in the rejection region and therefore we can conclude that the mean daily meat consumption can not be smaller than 65 kg. Now, let us calculate the power of the test. In a similar way the power function is

$$\beta(\mu) = P_\mu(\text{reject } H_0) = P(t_{35} < -1.6896 - \sqrt{n}(\mu - 65) / s_n)$$

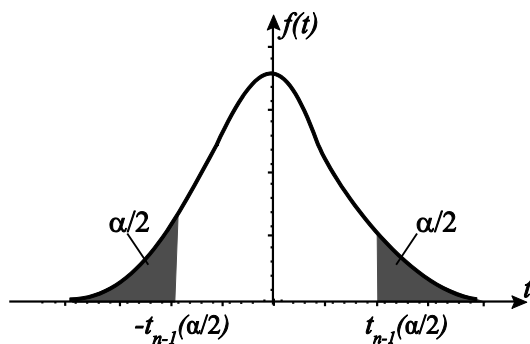
and the ampricial power (at the calculated mean value) is

$$\beta(63.5) = P(t_{35} < -1.6896 - \sqrt{36}(63.5 - 65) / \sqrt{42.4857}) = P(t_{35} < -0.3178) = 0.3763$$

As it is seen the powers are different even the actual variance being very close to the calculated variance.

c) $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$

Again we use the same value of the test statistic and reject the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_a : \mu \neq \mu_0$ if $|t_h| > t_{n-1}(\alpha/2)$ where the critical value can be found from t table as $P(t_{n-2} > t_{n-1}(\alpha/2)) = \alpha/2$. That is we reject the null hypothesis if $t_h > t_{n-1}(\alpha/2)$ or $t_h < -t_{n-1}(\alpha/2)$. The rejection region of the hypothesis testing problem is given in the following figure.



We reject the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_a : \mu \neq \mu_0$ if $t_h > t_{n-1}(\alpha/2)$ or $t_h < -t_{n-1}(\alpha/2)$ and the value of $t_{n-1}(\alpha/2)$ can be found in t table by using the significance level and the number of degrees of freedom.

The Rejection Region of the hypothesis testing of $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$

Example: Consider the lifetime of the refrigerator data given above. Assume that the lifetime of refrigerator is normally distributed with mean μ and variance σ^2 . The producer

wants to know the mean lifetimes of refrigerators in order to put a warranty on it. Based on randomly selected 25 refrigerator's lifetimes are given below table.

8.4	11.8	8.7	12.5	10.1	12.1	9.6	8.7	12.4	10.8	12.3	8.1	12.8
11.0	10.8	9.6	9.4	9.2	12.6	10.5	8.4	10.4	11.5	9.6	11.2	

Based on these observed values test whether the mean lifetime of the refrigerators is 10 years or not at 5% level. We want to test $H_0 : \mu = 10$ against the alternative $H_a : \mu \neq 10$

From the observed values we have calculated $\bar{x}_n = 10.5$ and $s_n^2 = 2.22$ remember that in the previous case we assume that the variance is known and it was 2.5) we calculate the value of the same test statistic based on this data. the value of the test statistic is

$$t_h = \sqrt{n}(\bar{x}_n - 10) / s_n = \sqrt{25}(10.5 - 10) / \sqrt{2.22} = 1.678.$$

The critical value of this hypothesis testing problem from the t table from the equation

$$P(t_{24} > t_{24}(\alpha / 2)) = 0.025$$

as $t_{24}(0.025) = 2.064$ and we reject the null hypothesis if $|t_h| > t_{24}(0.025)$ and since

$$|t_h| = 1.678 < 2.064 = t_{24}(0.025)$$

as a conclusion we fail to reject the null hypothesis. Again, the calculated variance (2.22) is very close to the actual variance (2.5) as we assume known variance and we calculated the empirical power (the power at the calculated mean 10.5) as 0.3861. The power function is similarly obtained as

$$\beta(\mu) = 1 - P_{\mu}(-2.064 - \sqrt{n}(\mu - 10) / s_n \leq t_{24} \leq -2.064 - \sqrt{n}(\mu - 10) / s_n).$$

Therefore, the empirical power of the test is

$$\begin{aligned} \beta(10.5) &= 1 - P(-2.064 - \sqrt{25}(10.5 - 10) / \sqrt{2.22} \leq t_{24} \leq -2.064 - \sqrt{25}(10.5 - 10) / \sqrt{2.22}) \\ &= 1 - P(-3.742 \leq t_{24} \leq 0.386) = 1 - [P(-3.742 \leq t_{24} \leq 0) + P(0 \leq t_{24} \leq 0.386)] \\ &= 1 - [P(0 \leq t_{24} \leq 3.742) + P(0 \leq t_{24} \leq 0.386)] = 1 - [0.5 + 0.1485] = 0.3515 \end{aligned}$$

As it is seen the powers are slightly changes even the actual variance being very close to the calculated variance.

CASE III: Hypothesis Testing for the normal population variance σ^2

In the following we are going to investigate statistical inferences about the normal population variance σ^2 . Again we will consider testing the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against different alternatives as

$$\text{a) } H_a : \sigma^2 > \sigma_0^2 \quad \text{b) } H_a : \sigma^2 < \sigma_0^2 \quad \text{c) } H_a : \sigma^2 \neq \sigma_0^2.$$

As we remember, if X_1, X_2, \dots, X_n is a random sample from a normal population with mean μ and variance σ^2 a reasonable estimator of the variance is the sample mean and

$$(n-1)S_n^2 / \sigma_0^2 \sim \chi_{n-1}^2.$$

If we want to test $H_0 : \sigma^2 = \sigma_0^2$ against any alternative we use a test statistic

$$K_h = (n-1)S_n^2 / \sigma_0^2$$

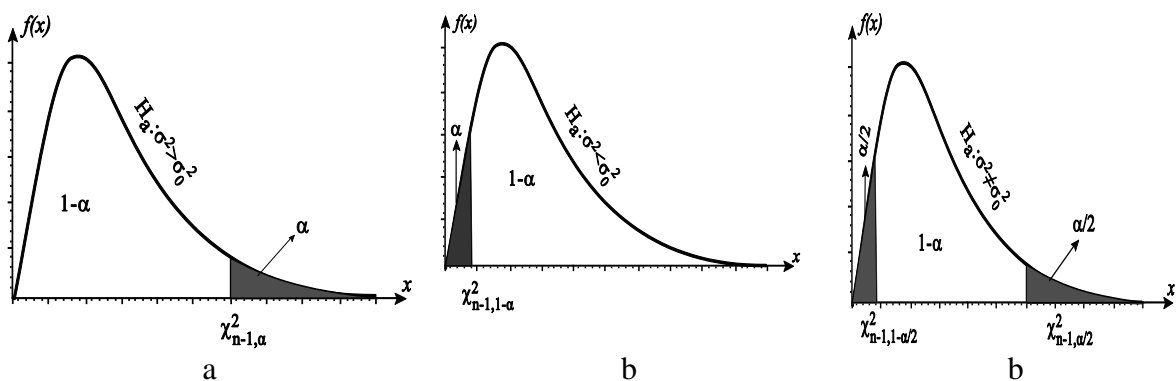
the critical values for various degrees of freedom have been calculated and they can be found in any statistical textbook. The rejection rules and regions for the following hypotheses

- a) $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 > \sigma_0^2$
- b) $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 < \sigma_0^2$
- c) $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 \neq \sigma_0^2$

are summarized below.

Hypothesis	Rejection Rule
$H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 > \sigma_0^2$	Reject H_0 if $K_h > \chi_{n-1, \alpha}^2$
$H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 < \sigma_0^2$	Reject H_0 if $K_h < \chi_{n-1, 1-\alpha}^2$
$H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 \neq \sigma_0^2$	Reject H_0 if $K_h < \chi_{n-1, 1-\alpha/2}^2$ or $K_h > \chi_{n-1, \alpha/2}^2$
Rejection rules for variance tests	

The rejection regions of the above hypothesis testing problems are given in the following figures



Rejection regions (if the calculated values fall within the shaded area we reject the null hypothesis)

Example: Consider the lifetime of refrigerator data given above.

8.4	11.8	8.7	12.5	10.1	12.1	9.6	8.7	12.4	10.8	12.3	8.1	12.8
-----	------	-----	------	------	------	-----	-----	------	------	------	-----	------

11.0	10.8	9.6	9.4	9.2	12.6	10.5	8.4	10.4	11.5	9.6	11.2	
------	------	-----	-----	-----	------	------	-----	------	------	-----	------	--

Based on the data we already calculated $s_n^2 = 2.22$ and in the previous example (in the case I) we assumed that $\sigma^2 = 2.5$. Let us try to test whether our assumption is valid or not.

Note that from chi-square table we find the critical values as 12.401 and 39.364 because

$$P(\chi_{24}^2 > 39.364) = 0.025 \quad \text{and} \quad P(\chi_{24}^2 < 12.401) = 0.025$$

and we calculate the value of the test statistic as

$$K_h = \frac{(n-1)s_n^2}{\sigma_0^2} = \frac{24(2.22)}{2.5} = 21.312.$$

As a decision rule, we are going to reject the null hypothesis $H_0 : \sigma^2 = 2.5$ against the alternative $H_a : \sigma^2 \neq 2.5$ at 5% level if $K_h < 12.401$ or $K_h > 39.364$ and since

$$12.401 = \chi_{n-1, 1-\alpha/2}^2 < K_h = 21.312 < \chi_{n-1, \alpha/2}^2 = 39.364$$

we fail to reject the null hypothesis and we can conclude that the variance is 2.5 at 5% level.

CASE IV Comparison of Normal Means

In application, the comparisons of normal means depend on the cases whether the variances of the populations are known or not. Let X_1, X_2, \dots, X_n be a random sample from a $N(\mu_x, \sigma_x^2)$ population and Y_1, Y_2, \dots, Y_m be another sample from $N(\mu_y, \sigma_y^2)$ population. Assume that these two populations are independent of each other. We want to compare the means of these two populations. Again we are going to test the following three hypotheses.

- a) $H_0 : \mu_x = \mu_y + a$ against $H_a : \mu_x > \mu_y + a$
or equivalently $H_0 : \mu_x \leq \mu_y + a$ against $H_a : \mu_x > \mu_y + a$
- b) $H_0 : \mu_x = \mu_y + a$ against $H_a : \mu_x < \mu_y + a$
or equivalently $H_0 : \mu_x \geq \mu_y + a$ against $H_a : \mu_x < \mu_y + a$
- c) $H_0 : \mu_x = \mu_y + a$ against $H_0 : \mu_x \neq \mu_y + a$

Again, we are going to investigate these hypotheses based on the cases whether the variances are known or not. First assume that the variances are known. In this case, testing $H_0 : \mu_x = \mu_y + a$ hypothesis is equivalent to $H_0 : \mu = a$ where $\mu = \mu_x - \mu_y$. A reasonable estimator of $\mu_x - \mu_y$ is $\bar{X}_n - \bar{Y}_m$. Notice that from the mean and the variance of $\bar{X}_n - \bar{Y}_m$

$$E(\bar{X}_n - \bar{Y}_m) = E(\bar{X}_n) - E(\bar{Y}_m) = \mu_x - \mu_y = \mu$$

and

$$\text{Var}(\bar{X}_n - \bar{Y}_m) = \text{Var}(\bar{X}_n) + \text{Var}(\bar{Y}_m) = (\sigma_x^2 / n) + (\sigma_y^2 / m)$$

we have the distribution of $\bar{X}_n - \bar{Y}_m$ as

$$\bar{X}_n - \bar{Y}_m \sim N(\mu_x - \mu_y, (\sigma_x^2 / n) + (\sigma_y^2 / m)) \text{ or } (\bar{X}_n - \bar{Y}_m) / \sqrt{(\sigma_x^2 / n) + (\sigma_y^2 / m)} \sim N(0,1).$$

The value of the test statistic $[(\bar{X}_n - \bar{Y}_m) - a] / \sqrt{(\sigma_x^2 / n) + (\sigma_y^2 / m)}$ will be denoted as

$$z_h = [(\bar{x}_n - \bar{y}_m) - a] / \sqrt{(\sigma_x^2 / n) + (\sigma_y^2 / m)}.$$

A) Assume the the variances of the both populations are known and consider the case $a = 0$. In the case,

1. We reject the null hypothesis $H_0 : \mu_x = \mu_y$ against the alternative $H_a : \mu_x > \mu_y$ at the significance level α if $z_h > z_\alpha$.

2. We reject the null hypothesis of $H_0 : \mu_x = \mu_y$ against the alternative $H_a : \mu_x < \mu_y$ at the significance level α if $z_h < -z_\alpha$.

3. We reject the null hypothesis of $H_0 : \mu_x = \mu_y$ against the alternative $H_a : \mu_x \neq \mu_y$ at the significance level α if $|z_h| > z_{\alpha/2}$.

B) Now assume that the variances of both populations are unknown and they are the same (say σ^2 , $\sigma^2 := \sigma_x^2 = \sigma_y^2$). The variance can be estimated anyone of the sample but since they are the same, we can use both sample (we have more observations, $n + m$ observations) called pooled variance. If we use the first sample, we estimate σ^2 by $S_{n,X}^2$ and if we use the second sample we estimate σ^2 by $S_{m,Y}^2$. However, it we combine these two samples in orderf to calculate the variance, we used the pooled variance calculated as

$$S_p^2 = \frac{(n-1)S_{n,X}^2 + (m-1)S_{m,Y}^2}{n+m-2}.$$

In order to test $H_0 : \mu_x = \mu_y$ against any alternatives givern above we calculate the t statistic as

$$t_h = \frac{[(\bar{x}_n - \bar{y}_m) - (\mu_x - \mu_y)]}{s_p \sqrt{(1/n) + (1/m)}}$$

1. We reject the null hypothesis of $H_0 : \mu_x = \mu_y$ against the alternative $H_a : \mu_x > \mu_y$ at the significance level α if $t_h > t_{n+m-2}(\alpha)$.

2. We reject the null hypothesis of $H_0 : \mu_x = \mu_y$ against the alternative $H_a : \mu_x < \mu_y$ at the significance level α if $t_h < -t_{n+m-2}(\alpha)$.

3. We reject the null hypothesis of $H_0 : \mu_x = \mu_y$ against the alternative $H_a : \mu_x \neq \mu_y$ at the significance level α if $|t_h| > t_{n+m-2}(\alpha/2)$.

Example: Assume that a statistic course has been teaching two different section by the same instructor. Assume also that the test score from both sections are normally distributed. The test scores for Section I is distributed as $N(\mu_x, \sigma_x^2)$ and the test scores for Section II are distributed as $N(\mu_y, \sigma_y^2)$. At the end of the semester an examination is given to both sections. From both classes 16 students are randomly selected and their test scores are given below.

Section I (X)								Section II (Y)							
60	65	60	70	75	80	65	69	70	72	65	64	50	62	67	66
83	78	63	67	69	73	79	67	49	84	73	67	48	66	63	72

Based on these test scores use $\alpha = 0.05$ and test whether the average for sections are the same or not against the alternative that in the average section I is better. That is, we want to test the null hypothesis $H_0 : \mu_x = \mu_y$ against $H_a : \mu_x > \mu_y$ at 5% level.

Based on the data some of the summary of statistics are calculated as

$$\sum_{i=1}^n x_i = 1123, \quad \sum_{i=1}^n x_i^2 = 79587, \quad \bar{x}_n = 70.1875, \quad s_{n,X}^2 = 51.095$$

$$\sum_{i=1}^m y_i = 1038, \quad \sum_{i=1}^m y_i^2 = 68682, \quad \bar{y}_m = 64.8750, \quad s_{m,Y}^2 = 89.45$$

and the pooled variance

$$s_p^2 = \frac{(n-1)s_{n,X}^2 + (m-1)s_{m,Y}^2}{n+m-2} = \frac{(16-1)s_{n,X}^2 + (16-1)s_{m,Y}^2}{(16+16)-2} = \frac{s_{n,X}^2 + s_{m,Y}^2}{2} = 70.27.$$

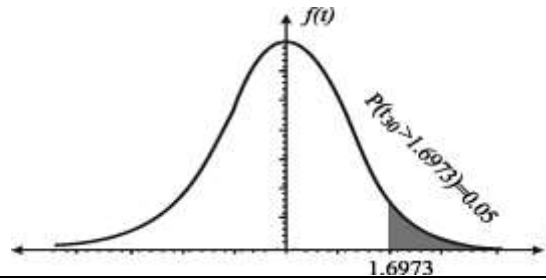
Assume that the variances of both populations are the same ($\sigma^2 := \sigma_x^2 = \sigma_y^2$). The value of the test statistic is calculated as

$$t_h = \frac{(\bar{x}_n - \bar{y}_m) - (\mu_x - \mu_y)}{s_p \sqrt{(1/n) + (1/m)}} = \frac{(70.1875 - 64.8750)}{8.38 \sqrt{(1/16) + (1/16)}} = \frac{\sqrt{16}(70.1875 - 64.8750)}{8.38 \sqrt{2}} \cong 1.8$$

and the critical value is found from t table for $\alpha = 0.05$ as $t_{30}(0.05) = 1.6973$. Since

$$t_h = 1.8 > 1.6973 = t_{30}(\alpha)$$

we reject the null hypothesis at 5% level. the critical region of the test is given in the following figure.



The Rejection Region of the hypothesis testing of $H_0 : \mu_x = \mu_y$ against $H_a : \mu_x > \mu_y$

Important Note: In the above discussion we assumed that the variances of the populations are the same. We can also test whether the variances are the same or not. that is we need to test the null hypothesis of $H_0 : \sigma_x^2 = \sigma_y^2$ against $H_a : \sigma_x^2 \neq \sigma_y^2$. In order to test this hypothesis we can use F test $F = S_{n,y}^2 / S_{m,y}^2$. If the variances are the same the value of F statistics is expected to be 1. However, if $S_{n,x}^2 < S_{n,y}^2$ then the value is always less than 1 and we fail to reject the null hypothesis. Therefore, we use

$$F = \frac{\max\{S_{n,x}^2, S_{m,y}^2\}}{\min\{S_{n,x}^2, S_{m,y}^2\}}$$

and we reject the null hypothesis $H_0 : \sigma_x^2 = \sigma_y^2$ if $F_h > F^{\alpha/2}(df_1, df_2)$ where df_1 represent the number of degrees of freedom for the numerator and df_2 represent the number of degrees of freedom for the denominator and

$$F_h = \frac{\max\{s_{n,x}^2, s_{m,y}^2\}}{\min\{s_{n,x}^2, s_{m,y}^2\}}$$

Remember that we calculated the variances as $s_{n,x}^2 = 51.095$ and $s_{m,y}^2 = 89.45$ and from the F table, we found the critical value as $F^{0.025}(15,15) = 2.862$. Since

$$F_h = \frac{\max\{s_{n,x}^2, s_{m,y}^2\}}{\min\{s_{n,x}^2, s_{m,y}^2\}} = \frac{\max\{51.095, 89.45\}}{\min\{51.095, 89.45\}} = \frac{89.45}{51.095} = 1.75 < 2.862 = F^{0.025}(15,15)$$

We fail to reject the null hypothesis of equal variances. That is, it is reasonable to assume that the variances of these two populations are the same.

Confidence Intervals:

In this part, we are going to write confidence intervals for the normal mean and variance. Confidence intervals are also known as “interval estimation” of the parameters. Again, we are going to have a random sample X_1, X_2, \dots, X_n from a $N(\mu, \sigma^2)$. There are going to be three

different cases in the confidence interval. Confidence intervals are very similar to two-sided hypothesis testing problems which we have studied above.

a) Confidence intervals for the normal mean μ when the variance is known

Let X_1, X_2, \dots, X_n be a random sample from a $N(\mu, \sigma^2)$ population. We want to write a $(1 - \alpha)100\%$ confidence interval for μ . For example if we want to write a 95% confidence interval for μ our significance level will be $\alpha = 0.05$. Note that the sample variance is also normally distributed with mean μ and variance σ^2 / n . That is, $\bar{X}_n \sim N(\mu, \sigma^2 / n)$. This implies that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$$

and therefore a $(1 - \alpha)100\%$ confidence interval can be written from the coverage probability as

$$1 - \alpha = P\left(-z_{\alpha/2} \leq \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq z_{\alpha/2}\right) = P\left(\bar{X}_n - \frac{\sigma}{\sqrt{n}}z_{\alpha/2} \leq \mu \leq \bar{X}_n + \frac{\sigma}{\sqrt{n}}z_{\alpha/2}\right).$$

Thus when the variance of the population is known, a $(1 - \alpha)100\%$ confidence interval for the normal mean μ based on the observed values x_1, x_2, \dots, x_n can be written as

$$\left(\bar{x}_n - \frac{\sigma}{\sqrt{n}}z_{\alpha/2}, \bar{x}_n + \frac{\sigma}{\sqrt{n}}z_{\alpha/2}\right)$$

where \bar{x}_n is the observed sample mean and $P(Z > z_{\alpha/2}) = \alpha / 2$.

b) Confidence intervals for the normal mean μ when the variance is unknown

If the variance of the population is unknown, we use its estimator S_n^2 and since

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \sim t_{n-1}$$

we use the following equality to write a $(1 - \alpha)100\%$ confidence interval for the normal mean μ from

$$\begin{aligned} 1 - \alpha &= P\left(-t_{n-1}(\alpha/2) \leq \frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \leq t_{n-1}(\alpha/2)\right) \\ &= P\left(\bar{X}_n - \frac{S_n}{\sqrt{n}}t_{n-1}(\alpha/2) \leq \mu \leq \bar{X}_n + \frac{S_n}{\sqrt{n}}t_{n-1}(\alpha/2)\right). \end{aligned}$$

Thus when the variance of the population is unknown, a $(1 - \alpha)100\%$ confidence interval for the normal mean μ based on the observed values x_1, x_2, \dots, x_n can be written as

$$\left(\bar{x}_n - \frac{s_n}{\sqrt{n}} t_{n-1}(\alpha/2), \bar{x}_n + \frac{s_n}{\sqrt{n}} t_{n-1}(\alpha/2) \right)$$

where \bar{x}_n and s_n^2 are observed sample mean and variance respectively and

$$P(t_{n-1} > t_{n-1}(\alpha/2)) = \alpha/2 .$$

c) Confidence intervals for the normal variance σ^2 .

We can also write confidence intervals for the normal variance by using the observed values. remember that if X_1, X_2, \dots, X_n is a random sample from $N(\mu, \sigma^2)$ we have

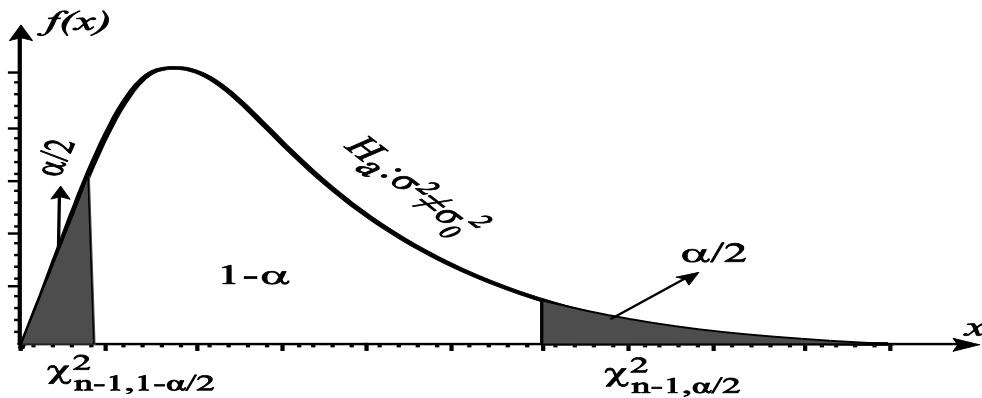
$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2 .$$

According to the following graph (testing for $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 \neq \sigma_0^2$) we can write a coverage probability as

$$1 - \alpha = P\left(\chi_{n-1, 1-\alpha/2}^2 \leq \frac{(n-1)S_n^2}{\sigma^2} \leq \chi_{n-1, \alpha/2}^2 \right) = P\left(\frac{(n-1)S_n^2}{\chi_{n-1, \alpha/2}^2} \leq \sigma^2 \leq \frac{(n-1)S_n^2}{\chi_{n-1, 1-\alpha/2}^2} \right)$$

and therefore based on the sample values x_1, x_2, \dots, x_n a $(1 - \alpha)100\%$ confidence interval for the normal variance is

$$\left(\frac{(n-1)s_n^2}{\chi_{n-1, \alpha/2}^2}, \frac{(n-1)s_n^2}{\chi_{n-1, 1-\alpha/2}^2} \right) .$$



Example: Assume that the test scores for a ststistic course is normally distributed with mean μ and variance σ^2 . A randomly selected 36 student's test scores are given below. Based on these observed values write a 95% confidence interval for μ and σ^2 .

75	73	81	56	62	65	55	58	59	61	65	68
72	77	69	66	60	64	58	63	67	84	53	52
55	57	62	66	75	72	76	60	65	75	62	70

Note that the variance is unknown. The estimated mean and the variance have been calculated as

$$\sum_{i=1}^{36} x_i = 2358, \quad \sum_{i=1}^{36} (x_i - \bar{x}_n)^2 = 2281, \quad \bar{x}_n = 65.5 \quad \text{and} \quad s_n^2 = 65.1714286 \cong 65.17.$$

We want to write 95% confidence intervals for μ and σ^2 and therefore $\alpha = 0.05$. From the t table, $P(t_{35} > t_{35}(0.025)) = 0.025$ implies that $t_{35}(0.025) = 2.03$ and from the chi-square table

$P(\chi_{35}^2 > \chi_{35}^2(0.025)) = 0.025$ we have $\chi_{35}^2(0.025) = 53.20$ and $P(\chi_{35}^2 > \chi_{35}^2(0.975)) = 0.975$ we find $\chi_{35}^2(0.975) = 20.57$. That is our critical values are

$$t_{35}(0.025) = 2.03, \quad \chi_{35}^2(0.025) = 53.20 \quad \text{and} \quad \chi_{35}^2(0.975) = 20.57.$$

Thus a 95% confidence interval for μ will be

$$\bar{x}_n \pm \frac{s}{\sqrt{n}} t_{35}(0.025) \quad \text{or} \quad 65.5 \pm \frac{8.073}{\sqrt{36}} (2.03) \quad \text{or} \quad 65.5 \pm 2.73 \quad \text{or} \quad (62.77, 68.23)$$

and a 95% confidence interval for σ^2 will be

$$\left(\frac{(n-1)s_n^2}{\chi_{n-1, \alpha/2}^2}, \frac{(n-1)s_n^2}{\chi_{n-1, 1-\alpha/2}^2} \right) \quad \text{or} \quad \left(\frac{(35)(65.17)}{53.20}, \frac{(35)(65.17)}{20.57} \right) \quad \text{or} \quad (42.875, 70.032).$$

Note: When we wanted to write a 95% confidence interval for the mean we assumed that the variance is unknown. If the variance was known (say $\sigma^2 = 64$ which is very close to calculated variance) and we wanted to write a confidence interval for the mean we had to use z table. In this case, since $P(Z > 1.96) = 0.025$ a confidence interval for the mean will be

$$\bar{x}_n \pm \frac{\sigma}{\sqrt{n}} z_{0.025} \quad \text{or} \quad 65.5 \pm \frac{8}{\sqrt{36}} (1.96) \quad \text{or} \quad 65.5 \pm 2.61 \quad \text{or} \quad (62.89, 68.11).$$