Method of separation variables (İ.Tosun, Modeling in Transport Phenomena , Elsevier, 2007)

Example :

Consider a rectangular slab of thickness 2L as shown in figure. Initally the concentration of A within the slab is uniform at a value of C_{Ao}. At t = 0 the surfaces at z = ± L are exposed to a fluid having a constant concentration of $C_{A\infty}$. Let us assume $Bi_{mass}>40$ so that resistance to mass transfer in the fluid phase is negligible and the concentration at the slab surfaces are almost equal to $C_{A\infty}$.

Introduce dimensionless quantities:

dimensionless concentration $\theta = \dfrac{C_{A\infty} - C_A}{C_{A\infty} - C_{Ao}}$

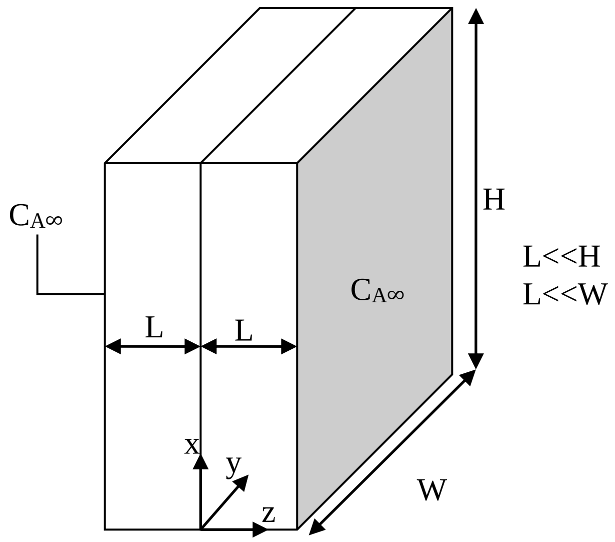
dimensionless location $\xi = \dfrac{z}{L}$

dimensionless time $\tau = \dfrac{t\,D_{AB}}{L^2}$

a) Derive the differential equation expressing the system and write initial and boundary conditions.

b) Solve the dimensionless equation by separation of variables

NOTE: If $\cos(\lambda) = 0$ then, $\lambda n = \left(n + \frac{1}{2}\right)\pi$; $A_n = \dfrac{2(-1)^n}{\left(n + \frac{1}{2}\right)\pi}$



Solution

a) $N_A HW\big|_z - N_A HW\big|_{z+\Delta z} = \frac{\partial}{\partial t}(C_A.V);\quad V = H.W.\Delta z$

$N_A.H.W\big|_z - N_A.H.W\big|_{z+\Delta z} = H.W.\Delta z \frac{\partial C_A}{\partial t}$

divide by $H.W.\Delta z$ and lim $\Delta z \to 0$,

$$-\frac{\partial(N_A)}{\partial z} = \frac{\partial C_A}{\partial t}$$

$$N_A = -D_{AB}\frac{\partial C_A}{\partial z}$$

$$D_{AB}\frac{\partial^2 C_A}{\partial z^2} = \frac{\partial C_A}{\partial t}$$

IC: $t = 0$ $C_A = C_{Ao}$
BC1: $z = L$ $C_A = C_{A\infty}$
BC2: $z = -L$ $C_A = C_{A\infty}$

b) dimensionless concentration $\theta = \frac{C_{A\infty} - C_A}{C_{A\infty} - C_{Ao}}$ $\partial\theta = -\frac{1}{C_{A\infty} - C_{Ao}}\partial C_A$

$$\partial C_A = -(C_{A\infty} - C_{Ao})\partial\theta$$

dimensionless location $\xi = \frac{z}{L}$ $\partial\xi = \frac{1}{L}\partial z$ $\partial z = L\partial\xi$

dimensionless time $\tau = \frac{t\,D_{AB}}{L^2}$ $\partial\tau = \frac{D_{AB}}{L^2}\partial t$ $\partial t = \frac{L^2}{D_{AB}}\partial\tau$

$$D_{AB}\frac{\partial^2 C_A}{\partial z^2}=\frac{\partial C_A}{\partial t}$$

$$D_{AB}\frac{\partial}{\partial z}\left(\frac{\partial C_A}{\partial z}\right)=\frac{\partial C_A}{\partial t}$$

$$D_{AB}\frac{\partial}{L\partial\xi}\left(\frac{-(C_{A\infty}-C_{Ao})\partial\theta}{L\partial\xi}\right)=\frac{-(C_{A\infty}-C_{Ao})\partial\theta}{\frac{L^2}{D_{AB}}\partial\tau}$$

$$\frac{\partial^2\theta}{\partial\xi^2}=\frac{\partial\theta}{\partial\tau}$$

IC: $\tau=0 \quad \theta=1$

BC1: $\xi=1 \quad \theta=0$

BC2: $\xi=-1 \quad \theta=0$

c) Separation of variables

$$\theta(\tau,\xi)=F(\tau)G(\xi)$$

$$\frac{\partial\theta}{\partial\tau}=G\frac{dF}{d\tau} \qquad \frac{\partial^2\theta}{\partial\xi^2}=F\frac{d^2G}{d\xi^2}$$

$$\frac{1}{G}\frac{d^2G}{d\xi^2}=\frac{1}{F}\frac{dF}{d\tau}=-\lambda^2$$

$$\frac{1}{F}\frac{dF}{d\tau}=-\lambda^2 \qquad F=I\exp\left(-\lambda^2\tau\right)$$

$$\frac{1}{G}\frac{d^2G}{d\xi^2}=-\lambda^2 \qquad G=c1\sin(\lambda\xi)+c2\cos(\lambda\xi)$$ Apply BC1 and BC2

BC1: $0=c1\sin(\lambda)+c2\cos(\lambda)$

BC2: $0=c1\sin(-\lambda)+c2\cos(-\lambda)$

$0=-c1\sin(\lambda)+c2\cos(\lambda)$

$c1\sin(\lambda)+c2\cos(\lambda)=-c1\sin(\lambda)+c2\cos(\lambda)$

$c1\sin(\lambda)=-c1\sin(\lambda)$

$c1=0$

$$\begin{array}{r}0=c1\sin(\lambda)+c2\cos(\lambda)\\ +\,0=-c1\sin(\lambda)+c2\cos(\lambda)\\ \hline 0=2c2\cos(\lambda)\end{array}$$

$\cos(\lambda)=0$

$$\lambda n = \left(n + \frac{1}{2} \right) \pi$$

$$F = I_n \exp\left(- \lambda_n{}^2 \tau \right)$$

$$G = c_n \cos(\lambda_n \xi)$$

$$\theta(\tau, \xi) = F(\tau) G(\xi) = \sum_{n=0}^{\infty} I_n \exp\left(-\lambda_n{}^2 \tau \right) c_n \cos(\lambda_n \xi)$$

$$A_n = I_n c_n$$

$$A_n = \frac{2(-1)^n}{\left(n + \frac{1}{2} \right) \pi}$$

$$\theta = \sum_{n=0}^{\infty} \frac{2(-1)^n}{\left(n + \frac{1}{2} \right) \pi} \exp\left(- \left(n + \frac{1}{2} \right)^2 \pi^2 \tau \right) \cos\left(\left(n + \frac{1}{2} \right) \pi \xi \right)$$

$$\theta = \frac{C_{A\infty} - C_A}{C_{A\infty} - C_{Ao}} = \sum_{n=0}^{\infty} \frac{2(-1)^n}{\left(n + \frac{1}{2} \right) \pi} \exp\left(- \left(n + \frac{1}{2} \right)^2 \pi^2 \tau \right) \cos\left(\left(n + \frac{1}{2} \right) \pi \xi \right)$$

$$C_A = C_{A\infty} - \frac{2(C_{A\infty} - C_{Ao})}{\pi} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n + \frac{1}{2} \right)} \exp\left(- \left(n + \frac{1}{2} \right)^2 \pi^2 \tau \right) \cos\left(\left(n + \frac{1}{2} \right) \pi \xi \right) \right]$$