Numerical solution of partial differential equations (Finite Difference)

Example :

Solve the one dimensional unsteady-state heat conduction equation by **finite differences method** with boundary conditions given below:

IC	$t = 0$	$T = 0°C$	for $0 \le x \le 10$
BC1	$x = 0$	$T = 100\ °C$ for all t	
BC2	$x = 10$ cm	$T = 0°C$	for all t

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \alpha = 2\ cm^2/s \quad \Delta t = 0.5\ s \qquad \Delta x = 2\ cm$$

<u>(Solve for t = 1.5 s)</u>

Solution :

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \alpha = 2\ cm^2/s \quad \Delta t = 0.5\ s \qquad \Delta x = 2\ cm$$

IC	$t = 0$	$T = 0°C$	for $0 \le x \le 10$
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$$u_{i,j+1} = r\,u_{i-1,j} + (1-2r)u_{i,j} + r\,u_{i+1,j}$$

$$r = \frac{ck}{h^2} \qquad \begin{aligned} c &= \alpha = 2\ cm^2/s \\ k &= \Delta t = 0.5\ s \\ h &= \Delta x = 2\,cm \end{aligned} \qquad r = \frac{(2)(0.5)}{(2)^2} = 0.25$$

$$\frac{B}{\Delta x} = \frac{10}{2} = 5$$

	i							
	j	0	1	2	3	4	5	6
t=0	**0**	100	0	0	0	0	0	0
t=0.5	**1**	100	25	0	0	0	0	0
t=1.0	**2**	100	37.5	6.25	0	0	0	0
t=1.5	**3**	100	50	12.5	1.56	0	0	0

$j = 0$
$i = 1, \dots, 5$

$$u_{1,1} = r\,u_{0,0} + (1-2r)u_{1,0} + r\,u_{2,0} = 0.25(100) + (1-0.5)0 + 0 = 25$$

$$u_{2,1} = r\,u_{1,0} + (1-2r)u_{2,0} + r\,u_{3,0} = 0.25(0) + (1-0.5)0 + 0 = 0$$

$j = 1$
$i = 1, \dots, 5$

$$u_{1,2} = r\,u_{0,1} + (1-2r)u_{1,1} + r\,u_{2,1} = 0.25(100) + (1-0.5)25 + 0 = 37.5$$

$$u_{2,2} = r u_{1,1} + (1-2r)u_{2,1} + r u_{3,1} = 0.25(25) + (1-0.5)0 + 0 = 6.25$$
$$u_{3,2} = r u_{2,1} + (1-2r)u_{3,1} + r u_{4,1} = 0 + (1-0.5)0 + 0 = 0$$

j = 2
i = 1, … , 5

$$u_{1,3} = r u_{0,2} + (1-2r)u_{1,2} + r u_{2,2} = 0.25(100) + (1-0.5)37.5 + 6.25 = 50$$
$$u_{2,3} = r u_{1,2} + (1-2r)u_{2,2} + r u_{3,2} = 0.25(37.5) + (1-0.5)6.25 + 0 = 12.5$$
$$u_{3,3} = r u_{2,2} + (1-2r)u_{3,2} + r u_{4,2} = 0.25(6.25) + (1-0.5)0 + 0 = 1.56$$
$$u_{4,3} = r u_{3,2} + (1-2r)u_{4,2} + r u_{5,2} = 0 + (1-0.5)0 + 0 = 0$$

j = 3
i = 1, … , 4

$$u_{1,4} = r u_{0,3} + (1-2r)u_{1,3} + r u_{2,3} = 0.5 + (1-1)0.75 + 0.25 = 0.75$$
$$u_{2,3} = r u_{1,2} + (1-2r)u_{2,2} + r u_{3,2} = 0.5 * 0.5 + (1-1)0.25 + 0 = 0.25$$
$$u_{3,3} = r u_{2,2} + (1-2r)u_{3,2} + r u_{4,2} = 0.5 * 0.25 + (1-1)0 + 0 = 0.125$$
$$u_{4,3} = r u_{3,2} + (1-2r)u_{4,2} + r u_{5,2} = 0 + (1-1)0 + 0 = 0$$