Ankara University
Department of Geological Engineering

GEO222 STATICS and STRENGTH of MATERIALS

Lecture Notes

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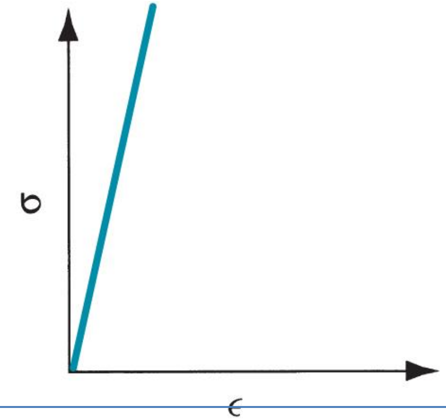
Typical Stress-Strain Behavior

Elastic Behavior

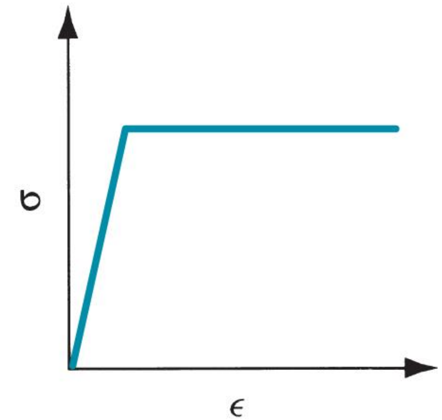
Behavior is defined completely by modulus of elasticity *E.* Fractures rather than yielding to plastic flow Brittle materials: ceramics, many cast irons, and thermosetting polymers



Perfectly Plastic Behavior

Stiffness defined by *E*

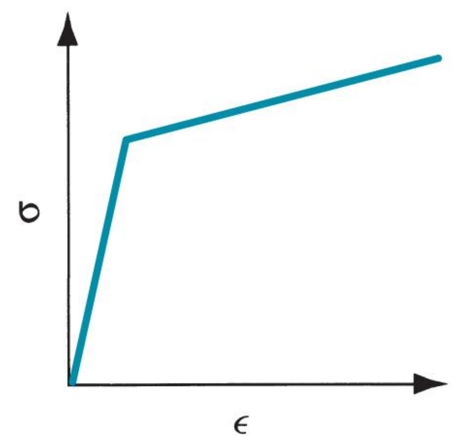
Once *Y* reached, deforms plastically at same stress level. Flow curve: $K = Y$, $n = 0$ Metals behave like this when heated to sufficiently high temperatures (above recrystallization)



Elastic and Strain hardening

Hooke's Law in elastic region, yields at *Y;* Flow curve: $K > Y$, $n > 0$

Most ductile metals behave this way when cold worked



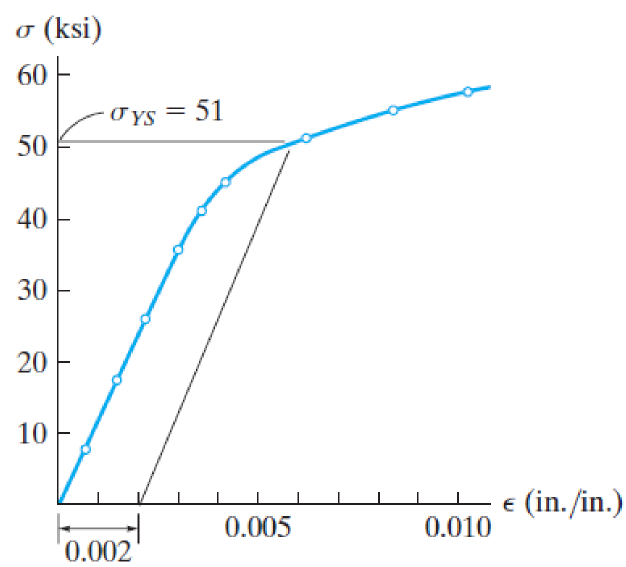
Stress–Strain Behavior of Ductile and Brittle Materials

Ductile Materials

Any material that can be subjected to large strains before it fractures is called a ***ductile material***. Mild steel is a typical example. Engineers often choose ductile materials for design because these materials are capable of absorbing shock or energy, and if they become overloaded, they will usually exhibit large deformation before failing. One way to specify the ductility of a material is to report its percent elongation or percent reduction in area at the time of fracture. The ***percent elongation*** is the specimen's fracture strain expressed as a percent. Thus, if the specimen's original gauge length is and its length at fracture is "L_f" then;

$$\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%) \qquad \text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%)$$

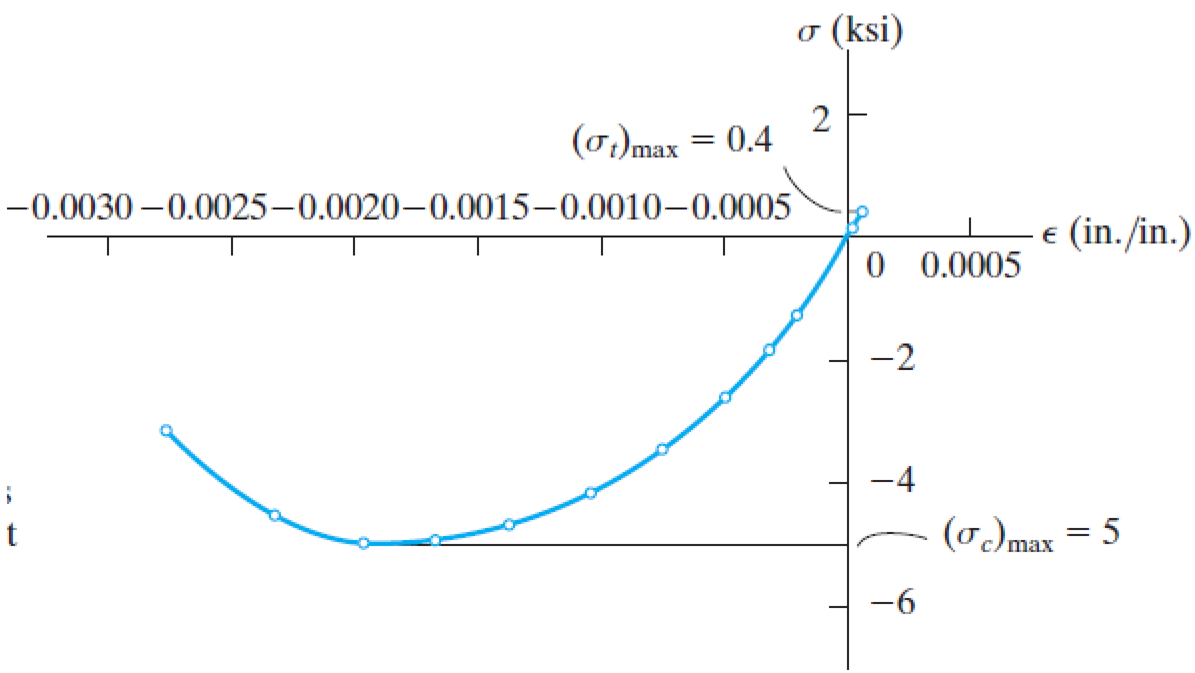
In most metals, however, constant yielding will *not occur* beyond the elastic range. One metal for which this is the case is aluminum. Actually, this metal often does not have a well-defined *yield point*, and consequently it is standard practice to define a ***yield strength*** using a graphical procedure called the ***offset method***. Normally a 0.2% strain is chosen, and from this point on the axis, a line parallel to the initial straight-line portion of the stress–strain diagram is drawn. The point where this line intersects the curve defines the yield strength.



Yield strength for an aluminum alloy

Brittle Materials

Materials that exhibit little or no yielding before failure are referred to as ***brittle materials***. Here fracture took place initially at an imperfection or microscopic crack and then spread rapidly across the specimen, causing complete fracture. Since the appearance of initial cracks in a specimen is quite random, brittle materials do not have a well-defined tensile fracture stress. Instead the *average* fracture stress from a set of observed tests is generally reported. Like gray cast iron, concrete is classified as a brittle material, and it also has a low strength capacity in tension. The characteristics of its stress–strain diagram depend primarily on the mix of concrete (water, sand, gravel, and cement) and the time and temperature of curing. A typical example of a "complete" stress–strain diagram for concrete is given below. Concrete is almost always reinforced with steel bars or rods whenever it is designed to support tensile loads.



$\sigma - \epsilon$ diagram for typical concrete mix

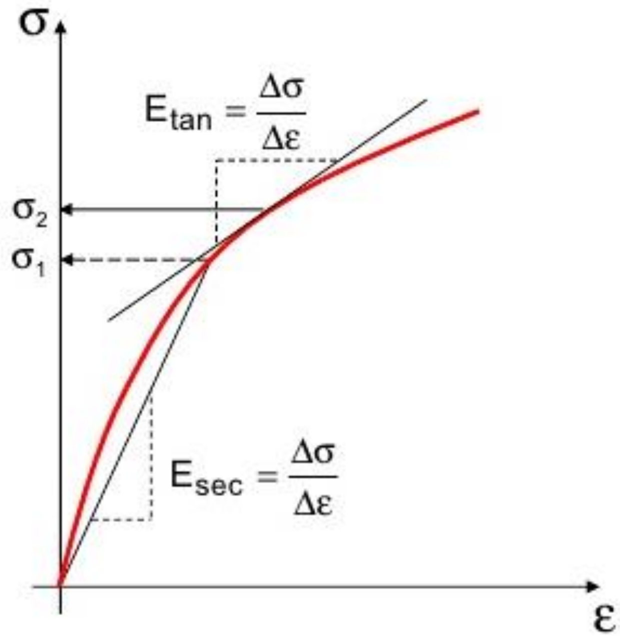
Hooke's Law

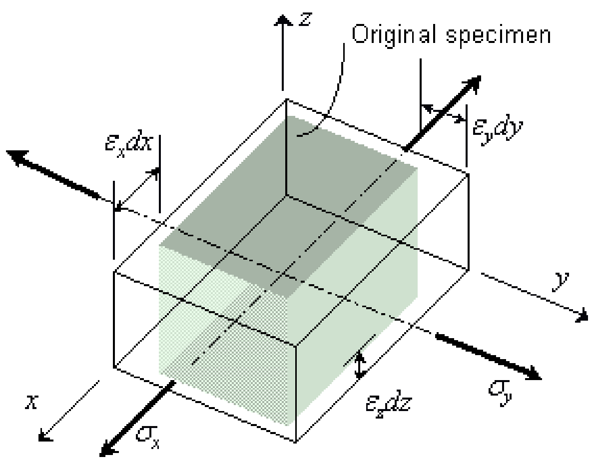
Hooke's law (1676) was first discovered using the extension and compression of springs. When applied to other materials, it was observed that most materials are also follow the same rule. Within the elastic region, an increase in stress causes a proportionate increase in strain. The "Modulus of Elasticity or Young's Modulus (Young, 1807)" is;

$$\sigma = E\varepsilon$$

σ: Stress
ε: strain
E:Modulus of elasticity

Hooke's law is valid through $0 \leq \sigma \leq \sigma_{yield}$





$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \qquad \epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \qquad \epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

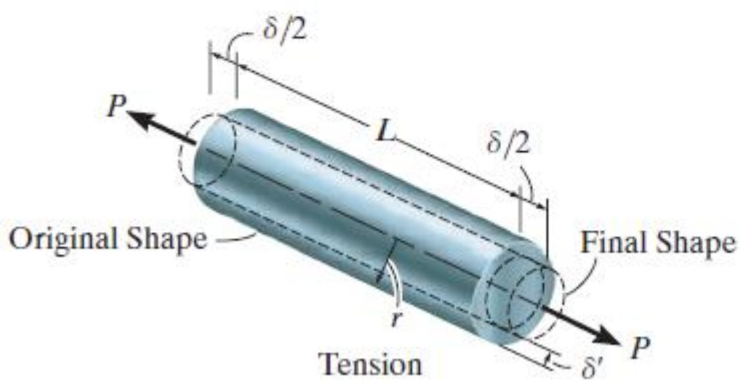
$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \qquad \epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

POISSON's RATIO

When a deformable body is subjected to an axial tensile force, not only does it elongate but it also contracts laterally. For example, if a rubber band is stretched, it can be noted that both the thickness and width of the band are decreased. Likewise, a compressive force acting on a body causes it to contract in the direction of the force and yet its sides expand laterally. Consider a bar having an original radius *r* and length *L* and subjected to the tensile force "*P*" given below. This force elongates the bar by an amount of (δ), and its radius contracts by an amount (δ'). Strains in the longitudinal or axial direction and in the lateral or radial direction are respectively,

$$\epsilon_{long} = \frac{\delta}{L} \quad \text{and} \quad \epsilon_{lat} = \frac{\delta'}{r}$$

Poisson's ratio (Poisson, 1818) realizes that, within the elastic range, the ratio of strain is constant, since the deformations (δ and δ')are proportional. Poisson's ratio (υ) has a numerical value that is unique for a particular material which is homogenous and isotropic. Negative sign indicates the negative strain (lateral contraction) due to the longirudinal elongation (positive strain). Typical values are between ¼ and 1/3. For an ideal material with no deformation, the value is "0". Also, based on numerous test data, the maximum value is 0.5.



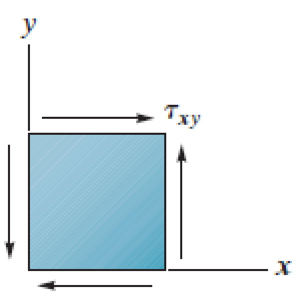
Shear Stress-Strain Diagram

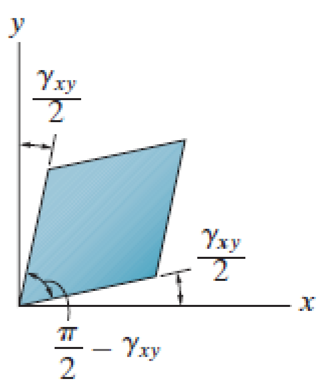
When a small element of material is subjected to "*pure shea"r*", equilibrium requires that equal shear stresses must be developed on four faces of the element. These stresses must be directed toward or away from diagonally opposite corners of the element, as shown in Fig.a. If the material is *homogeneous* and *isotropic*, then this shear stress will distort the element uniformly (Fig.*b*) Shear strain (γ_{xy}) measures the angular distortion of the element relative to the sides originally along the *x* and *y* axes. Like the tension test, this material when subjected to shear will exhibit linear-elastic behavior and it will have a defined "*proportional limit*" (τ_{pl}). Also, strain hardening will occur until an "*ultimate shear stress*" (τ_u) is reached. And finally, the material will begin to lose its shear strength until it reaches a point where it fractures (τ_f). For most engineering materials, like the one just described, the elastic behavior is *linear*, and so Hooke's law for shear can be written as;

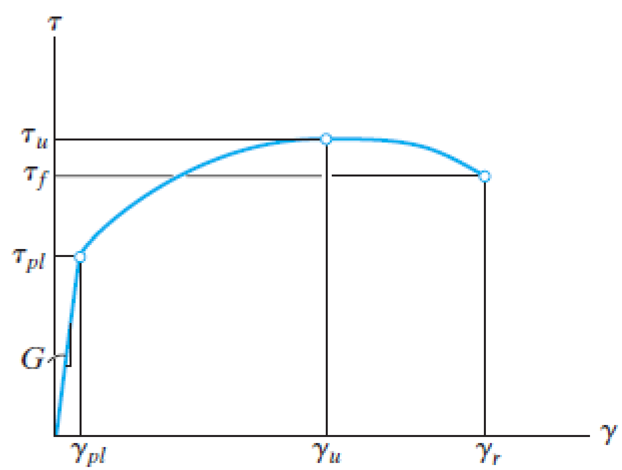
$$\tau = G\gamma$$

G: Shear modulus (Modulus of rigidity) which corresponds to the slope τ-γ diagram ($G=\tau_{pl}/\gamma_{pl}$).

$$G = \frac{E}{2(1+\upsilon)}$$

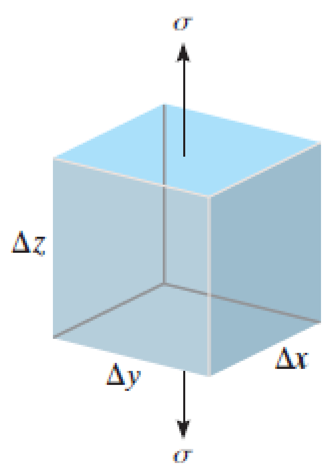






Strain Energy

As the material is deformed, an internally stored energy occurs and directly related to strain development. If the element below is subjected to loading, stress will develop an additional force ΔF=σΔA=σ(ΔxΔy) on the top and bottom faces of the element. Due, a vertical displacement develops (εΔz). The work during the process is equivalent to the strain energy stored in the element assuming no energy is lost in the form of heat. Strain energy is expressed as the energy per volume of the loaded element.



$$\Delta U = \left(\frac{1}{2}\Delta F\right)\varepsilon\Delta z = (\frac{1}{2}\sigma\Delta x\Delta y)\varepsilon\Delta z$$

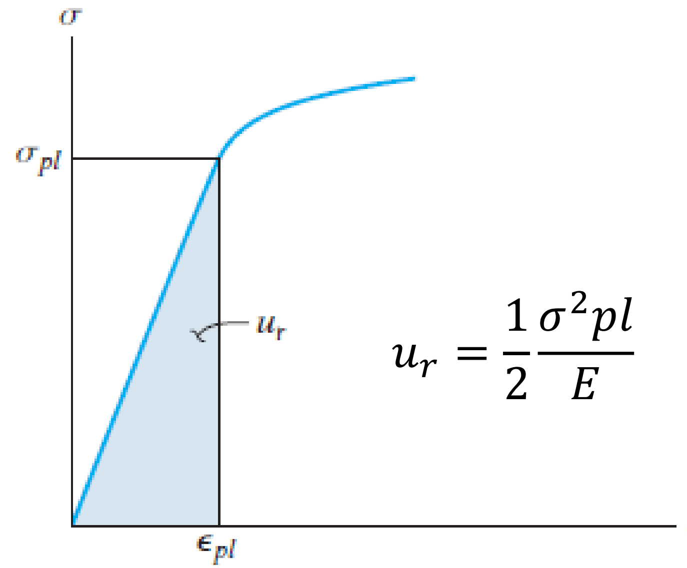
$$\Delta U = (\frac{1}{2}\sigma\varepsilon\Delta \mathrm{V})$$

If the material is linear-elastic, Hooke's law is applied and strain energy can be expressed in terms of stress.

$$U = (\frac{1}{2}\frac{\sigma^2}{E})$$

Modulus of Resilience

When the stress reaches proportional limit, the strain energy density is equal to the modulus of resilience. The modulus is the triangular area under the elastic region of the stress-strain diagram. Resilience represents the ability of materials to absorb energy without permanent damage.



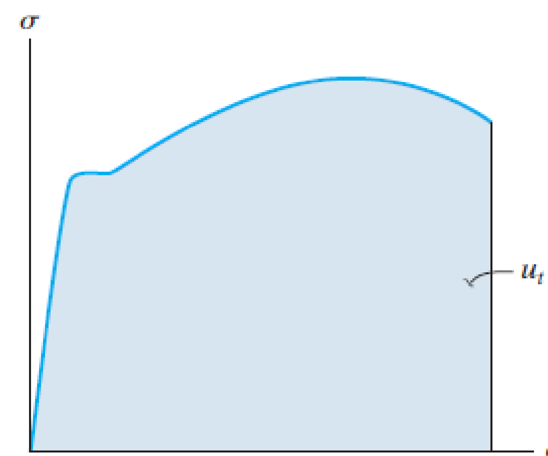
$$u_r = \frac{1}{2}\frac{\sigma^2 pl}{E}$$

Modulus of resilience u_r

Modulus of Toughness

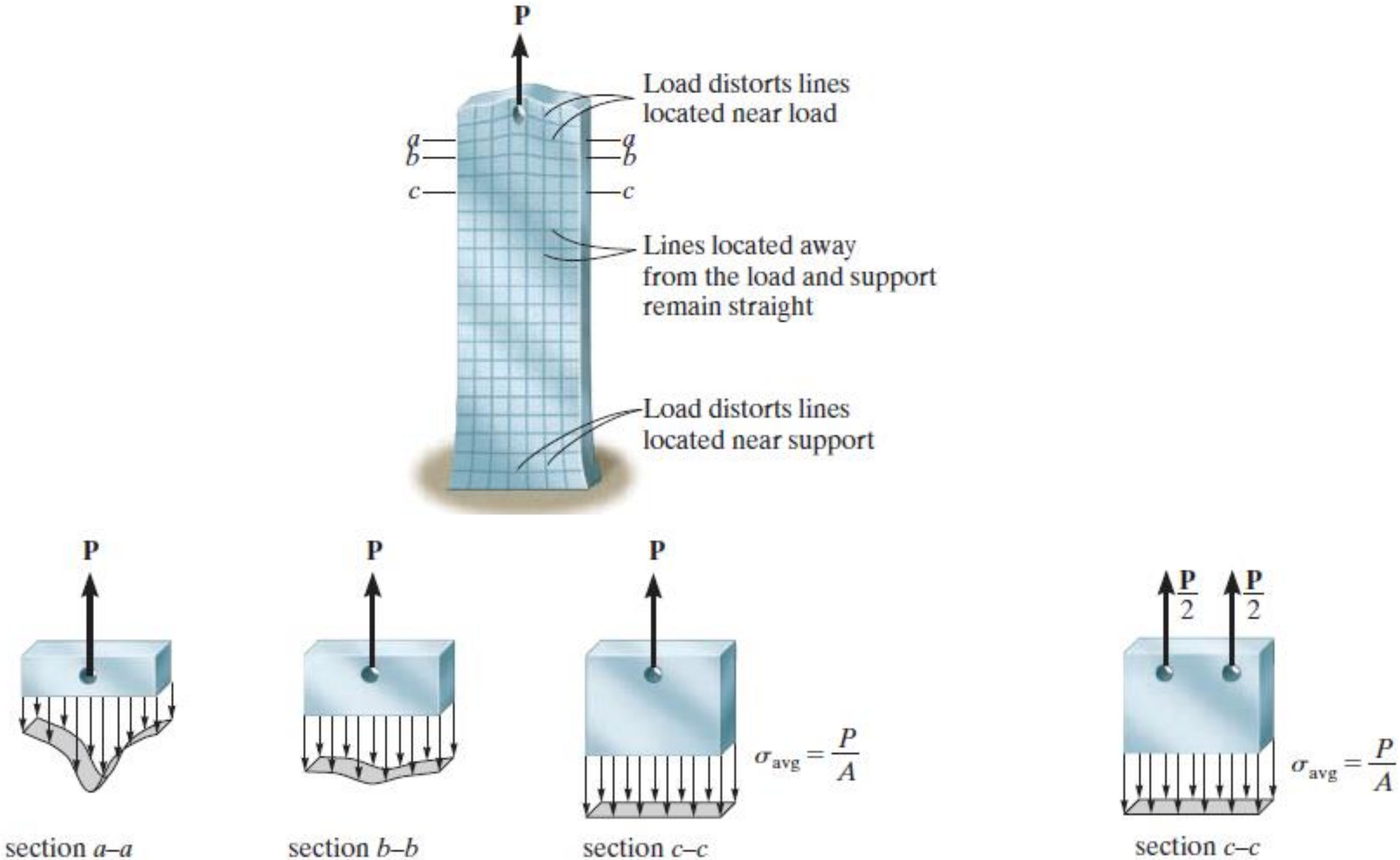
Another important property of a material is the ***modulus of toughness***, This quantity represents the *entire area* under the stress–strain diagram and therefore it indicates the strain-energy density of the material just before it fractures.

This property becomes important when designing members that may be accidentally overloaded. Alloying metals can also change their resilience and toughness. For example, by changing the percentage of carbon in steel, the resulting stress–strain diagrams how the degrees of resilience and toughness can be changed.



SAINT-VENANT'S PRINCIPLE

Essentially it states that the *stress and strain produced at points in a body sufficiently removed from the region of load application will be the same as the stress and strain produced by any applied loadings that have the same statically equivalent resultant, and are applied to the body within the same region*. For example, if two symmetrically applied forces "P/2" act on the bar, stress distribution at section *c–c* will be uniform and therefore equivalent to $\sigma_{avg} = P/A$.



(b)

(c)

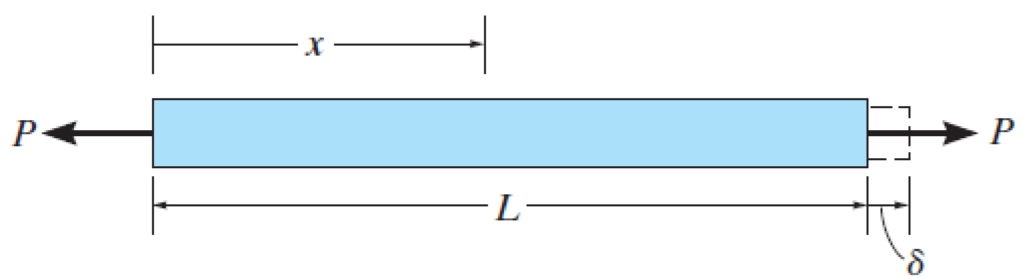
Elastic Deformation of an Axially Loaded Member

In many cases the bar will have a constant cross-sectional area *A*; and the material will be homogeneous, so *E* is constant. Furthermore, if a constant external force is applied at each end, then the internal force *P* throughout the length of the bar is also constant.

$$\delta = \frac{PL}{AE}$$

If the bar is subjected to several different axial forces along its length, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next, the above equation can be applied to each *segment* of the bar where these quantities remain *constant*. The displacement of one end of the bar with respect to the other is then found from the *algebraic addition* of the relative displacements of the ends of each segment. For this general case,

$$\delta = \sum \frac{PL}{AE}$$



Statically Indeterminate Axial Loading

Consider the bar shown in Fig. 4–11*a* which is fixed supported at both of its ends. From the free-body diagram, Fig. 4–11*b*, equilibrium requires

$$+\uparrow \Sigma F = 0; \qquad F_B + F_A - P = 0$$

This type of problem is called ***statically indeterminate***, since the equilibrium equation(s) are not sufficient to determine the two reactions on the bar.

In order to establish an additional equation needed for solution, it is necessary to consider how points on the bar displace. Specifically, an equation that specifies the conditions for displacement is referred to as a ***compatibility*** or ***kinematic condition***. In this case, a suitable compatibility condition would require the displacement of one end of the bar with respect to the other end to be equal to zero, since the end supports are fixed. Hence, the compatibility condition becomes

$$\delta_{A/B} = 0$$

This equation can be expressed in terms of the applied loads by using a *load–displacement relationship*, which depends on the material behavior. For example, if linear-elastic behavior occurs, $\delta = PL/AE$ can be used. Realizing that the internal force in segment AC is $+F_A$, and in segment CB the internal force is $-F_B$, Fig. 4–11*c*, the above equation can be written as

$$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0$$

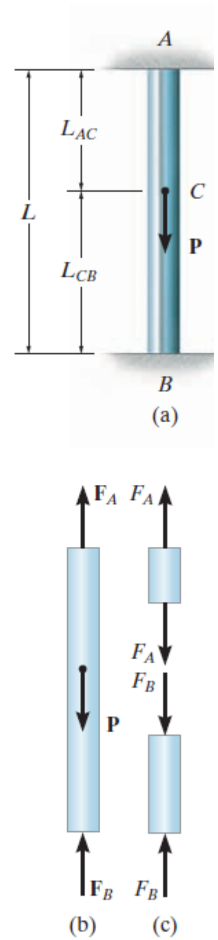


Fig. 4–11

Assuming that AE is constant, then $F_A = F_B(L_{CB}/L_{AC})$, so that using the equilibrium equation, the equations for the reactions become

$$F_A = P\left(\frac{L_{CB}}{L}\right) \quad \text{and} \quad F_B = P\left(\frac{L_{AC}}{L}\right)$$

Since both of these results are positive, the direction of the reactions is shown correctly on the free-body diagram.

Force Method of Analysis

It is also possible to solve statically indeterminate problems by writing the compatibility equation using the principle of superposition. This method of solution is often referred to as the ***flexibility or force method of analysis***. To show how it is applied, consider again the bar in Fig. 4–16*a*. If we choose the support at *B* as "redundant" and *temporarily* remove its effect on the bar, then the bar will become statically determinate as in Fig. 4–16*b*. By using the principle of superposition, we must add back the unknown redundant load $\mathbf{F}_B$, as shown in Fig. 4–16*c*.

If load **P** causes *B* to be displaced *downward* by an amount δ_P, the reaction $\mathbf{F}_B$ must displace end *B* of the bar *upward* by an amount δ_B, such that no displacement occurs at *B* when the two loadings are superimposed. Thus,

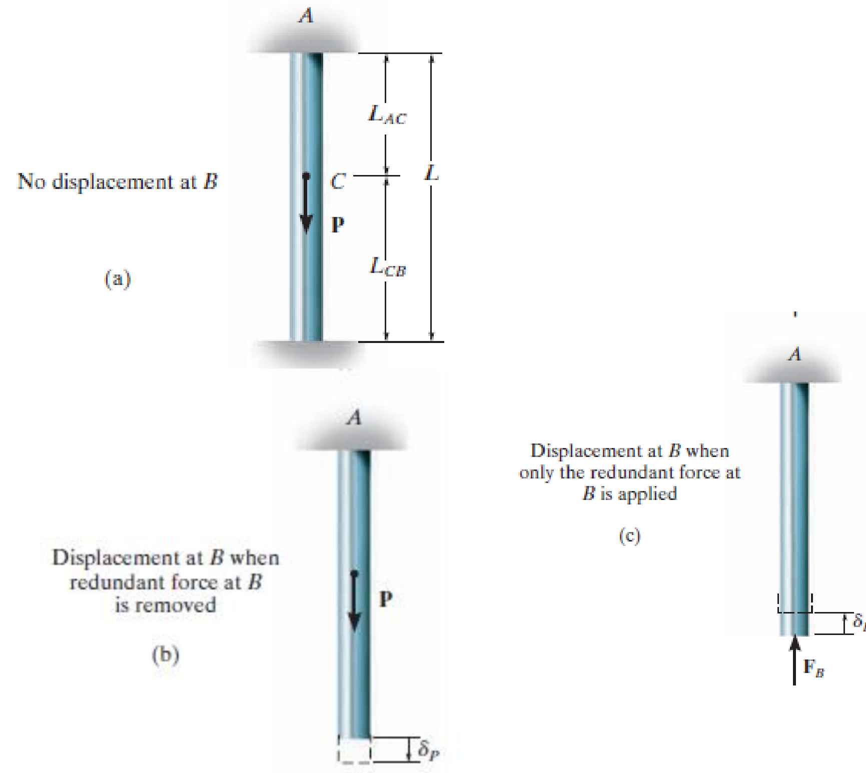
$$(+\downarrow) \qquad 0 = \delta_P - \delta_B$$

This equation represents the compatibility equation for displacements at point *B*, for which we have assumed that displacements are positive downward.

Applying the load–displacement relationship to each case, we have $\delta_P = PL_{AC}/AE$ and $\delta_B = F_B L/AE$. Consequently,

$$0 = \frac{PL_{AC}}{AE} - \frac{F_B L}{AE}$$

$$F_B = P\left(\frac{L_{AC}}{L}\right)$$



From the free-body diagram of the bar, Fig. 4–11*b*, the reaction at *A* can now be determined from the equation of equilibrium,

$$+\uparrow \Sigma F_y = 0; \qquad P\left(\frac{L_{AC}}{L}\right) + F_A - P = 0$$

Since $L_{CB} = L - L_{AC}$, then

$$F_A = P\left(\frac{L_{CB}}{L}\right)$$

These results are the same as those obtained in Sec. 4.4, except that here we have applied the condition of compatibility to obtain one reaction *and then* the equilibrium condition to obtain the other.