NUCLEAR REACTIONS

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NUCLEAR REACTIONS

If energetic particles from a reactor or accelerator (or even from a radioactive source) are allowed to fall upon an element, there is the possibility of a nuclear reaction taking place. The first such nuclear reactions were done in Rutherford's laboratory, using α particles from a radioactive source.

Rutherford was able to observe a change or transmutation of nuclear species in the reaction done in 1919:

$$\alpha + {}^{14}_{7}N \rightarrow {}^{17}_{8}O + p$$

The first particle accelerator capable of inducing nuclear reactions was built by Cockcroft and Walton, who in 1930 observed the reaction

$$p + {}^{7}_{3}Li \rightarrow {}^{4}_{2}He + \alpha$$

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The nucleus formed after bombardment is, in most cases, different from the target nucleus. Such a change of the target nucleus is a transmutation and the reaction itself is called a transmutation reaction.

The importance of the study of nuclear reactions lies in the fact that most of the information about the properties of the nucleus (such as the size, the charge distribution, and the nature of the nuclear forces) is obtained from these investigations. A typical nuclear reaction is written as

$$x + X \rightarrow Y + y \qquad (1)$$

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where x is the accelerated particle, X is the target (usually stationary in the laboratory), and Y and y are the reaction products. Usually, Y will be a heavy product that stops in the target and is not directly observed, while y is a light particle that can be detected and measured. Generally, x and y will be nucleons or light nuclei, but occasionally y will be a γ ray, in which case the reaction is called *radiative capture*.

An alternative and compact way of indicating the nuclear reaction represented by Eq. 1 is

$$X\left(x, y\right)Y \qquad (2)$$

We are interested in the study of nuclear reactions from two view points:

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A: The conditions under which different reactions take place. In many case it is possible to predict the outcome of a nuclear reaction, but we shall limit ourselves to looking into the conditions necessary to start a nuclear reaction.

B: The determination of the probability of an incoming particle being absorbed by the target nucleus. This probability is called the cross section of a given nuclear reaction.

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

Let us assume that x and X are far apart and do not exert any force on each other. This implies that the system does not have any potential energy.

Let us say that long before the collision between the incoming particle, x, and the target nucleus, X, their rest masses are m_x and M_X, and their kinetic energies are K_x and K_X, respectively.

Thus the total energy, E_i of the initial system

$$E_i = K_x + m_x c^2 + K_X + M_X c^2 \qquad (3)$$

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

Similarly the final energy, E_f, of the system $Y+y$, long after the collision, is

$$E_f = K_Y + M_Y c^2 + K_y + m_y c^2 \qquad (4)$$

Because there are no external forces acting on the system, the final energy must be equal to the initial energy

$$E_f = E_i \qquad (5)$$

or

$$K_Y + M_Y c^2 + K_y + m_y c^2 = K_x + m_x c^2 + K_X + M_X c^2 \qquad (6)$$

$$\left[\left(K_Y + K_y\right) - \left(K_X + K_x\right)\right] = \left[\left(M_X + m_x\right)c^2 - \left(M_Y + m_y\right)c^2\right] \qquad (7)$$

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

The net change in the kinetic energy is called the *disintegration energy* or *Q-value* of the nuclear reaction,

$$Q = \left(K_Y + K_y\right) - \left(K_X + K_x\right) \qquad (8)$$

Q is also equal to the change in the rest-mass energies given by

$$Q = \left(M_X + m_x\right)c^2 - \left(M_Y + m_y\right)c^2 \qquad (9)$$

The Q value may be positive, negative, or zero. If $Q>0$ ($m_{initial} > m_{final}$) or $K_{final} > K_{initial}$) the reaction is said to be exoergic or exothermic; in this case nuclear mass or binding energy is released as kinetic energy of the final products.

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

When $Q<0$ ($m_{initial} < m_{final}$) or $K_{final} < K_{initial}$) the reaction is said to be endoergic or endothermic, and initial kinetic energy is converted into nuclear mass or binding energy.

The changes in mass and energy must of course be related by the familiar expression from special relativity, $\Delta \boldsymbol{E} = \Delta mc^2$ - any change in the kinetic energy of the system of reacting particles must be balanced by an equal change in its rest energy.

In most experiments the target nucleus is initially at rest. In such cases

$$Q = \left(K_Y + K_y\right) - K_x$$

$$= \left(M_X + m_x\right)c^2 - \left(M_Y + m_y\right)c^2 \qquad (10)$$

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

In general, it is not easy to measure accurately the kinetic energy, K_Y, of the recoil nucleus. If we consider the conservation of momentum, it is possible to obtain an expression for the *Q-value* independent of K_Y.

Consider a particle x of mass m_x moving with velocity v_x that strikes the target nucleus, X, of mass M_X and whose velocity is zero, i.e., at rest. After the nuclear reaction, the recoil nucleus Y makes an angle ϕ with the initial direction of x and has mass M_Y and velocity V_Y, while the particle y makes an angle θ and has mass m_y and velocity v_y.

(See Fig. 4.1 in page 82 in Fundamentals of Nuclear Physics, by Atam P. Arya)

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

From the conservation of momentum, we get

$$m_x v_x = m_y v_y \cos\theta + M_Y V_Y \cos\phi$$

$$0 = m_y v_y \sin\theta - M_Y V_Y \sin\phi$$

or

$$M_Y V_Y \cos\phi = m_x v_x - m_y v_y \cos\theta \qquad (11)$$

$$M_Y V_Y \sin\phi = m_y v_y \sin\theta \qquad (12)$$

Squaring and adding Eqs 11 and 12, we get

$$M_Y{}^2 V_Y{}^2 = m_x{}^2 v_x{}^2 + m_y{}^2 v_y{}^2 - 2 m_x m_y v_x v_y \cos\theta \qquad (13)$$

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

Making use of the relations,

$$K_x = \frac{1}{2} m_x v_x^{\ 2}, \quad K_y = \frac{1}{2} m_y v_y^{\ 2} \quad and \quad K_Y = \frac{1}{2} M_Y V_Y^{\ 2}$$

in Eq. 13 and rearranging the terms, we get

$$K_Y = \frac{m_x}{M_Y} K_x + \frac{m_y}{M_Y} K_y - \frac{2}{M_Y} \left(m_x m_y K_x K_y \right)^{1/2} \cos\theta \qquad (14)$$

The *Q-value* of the reaction with $K_X = 0$ is given by Eq. 10

$$Q = \left(K_Y + K_y \right) - K_x$$

and substituting the value of K_Y from Eq. 14 into Eq. 10

$$Q = K_y \left(1 + \frac{m_y}{M_Y} \right) - K_x \left(1 - \frac{m_x}{M_Y} \right) - \frac{2}{M_Y} \left(m_x m_y K_x K_y \right)^{1/2} \cos\theta \qquad (15)$$

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

Eq. 15 is the general equation for the Q-value of a nuclear reaction. If the outgoing particles are observed at right angles to the direction of the incoming particles, i.e., $\theta=90^0$, $\cos 90^0 = 0$ then Eq. 15 becomes

$$Q = K_y\left(1+\frac{m_y}{M_Y}\right)-K_x\left(1-\frac{m_x}{M_Y}\right) \qquad (16)$$

This is equivalent to the case in which the target nucleus and, hence, the recoil nucleus are of infinite mass. Rewriting Eq.15, we can express the kinetic energy of the outgoing particle in the following form,

$$\left(M_Y+m_y\right)K_y-2\left(m_x m_y K_x\right)^{1/2}\cos\theta\sqrt{K_y}-\left[K_x\left(M_Y-m_x\right)+M_Y Q\right]=0$$

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

Which is a quadratic in $\sqrt{K_y}$

Solving it we get

$$\sqrt{K_y} = \frac{\sqrt{m_x m_y K_x}\cos\theta \pm \left\{\left(m_x m_y K_x \cos^2\theta\right) + \left(M_Y + m_y\right)\left[K_x\left(M_Y - m_x\right) + M_Y Q\right]\right\}^{1/2}}{\left(M_Y + m_y\right)} \qquad (17)$$

or

$$\sqrt{K_y} = a \pm \sqrt{a^2 + b} \qquad (18)$$

where

$$a = \frac{\sqrt{m_x m_y K_x}}{\left(M_Y + m_y\right)}\cos\theta \qquad \text{and} \qquad b = \frac{K_x\left(M_Y - m_x\right) + M_Y Q}{\left(M_Y + m_y\right)}$$

ENERGY CONSERVATIONS IN NUCLEAR REACTIONS

If the bombarding energy is almost zero, i.e., $K_x \simeq 0$, which happens in the case of reactions initiated by the capture of thermal neutrons, Eq. 17 reduces to

$$K_y = \frac{M_Y Q}{\left(M_Y + m_y\right)} \qquad for \quad Q \rangle 0$$

If $Q>0$ and $M_Y > m_x$, only one of the two solutions of K_y, obtained from Eq. 17 and 18, will be positive, and is given by

$$\sqrt{K_y} = a + \sqrt{a^2 + b}$$

In this case K_y does depend on the angle θ. K_y has a maximum value for $\theta=0$, minimum for $\theta=180^0$, and for $\theta=90^0$, K_y =b

$$K_y = \frac{K_x\left(M_Y - m_x\right) + M_Y Q}{\left(M_Y + m_y\right)}$$

THRESHOLD ENERGY FOR AN ENDOERGIC REACTION

We derived the expression for the *Q-value* of a nuclear reaction by considering the reaction to be taking place in the LAB coordinate system. The endoergic reaction is one for which the *Q-value* is negative. In this case some of the initial kinetic energy (equal to the *Q-value* of the reaction) is converted into the rest-mass energy of the final products.

The minimum value of the energy required for an endoergic reaction to take place is the threshold energy.

For almost zero bombarding energies, $K_x \simeq 0$, we get

$a \simeq 0$, and $b \simeq M_Y \mathrm{Q}/(\mathrm{M}_Y + \mathrm{m}_\mathrm{y})$

THRESHOLD ENERGY FOR AN ENDOERGIC REACTION

and because *Q is negative*, the quantity (a^2+b) is negative. This means that $\sqrt{K_y}$ is an imaginary quantity, or K_y is negative, which does not have any physical meaning. Thus the endoergic reactions are not possible with this insufficient amount of kinetic energy.

As the energy K_x of the bombarding particle is increased, the reaction will become possible with a certain minimum value of K_x given by the condition $a^2+b = 0$

$$\left(K_x\right)_\theta = -Q\left[\frac{M_Y + m_y}{\left(M_Y + m_y - m_x - \left(\frac{m_x m_y}{M_Y}\right)\sin^2\theta\right)}\right]$$

THRESHOLD ENERGY FOR AN ENDOERGIC REACTION

If the outgoing particle of mass m_y is observed at $\theta=0^0$, this leads to

$$\left(K_x\right)_{\min} = -Q\left[\frac{M_Y + m_y}{M_Y + m_y - m_x}\right]$$

Using the relation

$$M_X + m_x = M_Y + m_y + \frac{Q}{c^2}$$

we get

$$\left(K_x\right)_{\min} = -Q\left[\frac{M_X + m_x - \frac{Q}{c^2}}{M_X - \frac{Q}{c^2}}\right] \qquad (18)$$

THRESHOLD ENERGY FOR AN ENDOERGIC REACTION

Because the energy equivalent of the mass, M_X, is usually very large as compared to Q, we may write Eq. 18 as

$$(K_x)_{\min} = -Q\left[\frac{M_X + m_x}{M_X}\right] = -Q\left(1 + \frac{m_x}{M_x}\right)$$

$$Threshold\ Energy = (K_x)_{\min} = |Q|\left(1 + \frac{m_x}{M_x}\right) \qquad (19)$$

Thus we conclude that if the energy of the incident particles is equal to the threshold energy, the outgoing particles are emitted only in the direction θ=0 with energy given by the following

$$K_y = (K_x)_{threshold} \frac{m_x m_y}{(m_y + M_Y)^2}$$

THRESHOLD ENERGY FOR AN ENDOERGIC REACTION

If the reaction reaches excited states of Y, the Q-value equation should include the mass energy of the excited state.

$$Q_{ex} = \left(M_X + m_x - M_{Y^*} - m_y\right)c^2$$
$$= Q_0 - E_{ex}$$

Where Q_0 is the *Q-value* corresponding to the ground state of *Y*, and where we have used

$$M_{Y^*}c^2 = M_Y c^2 + E_{ex}$$

as the mass energy of the excited state (E_{ex} is the excitation energy above the ground state)

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

In the previous section, the laboratory coordinate system (LAB coordinate system) was used to explain the dynamics of nuclear reactions. But it is usually more convenient from the theoretical view point to use the center-of-mass coordinate system (CMCS). CMCS can be utilized to calculate the minimum energy required by the bombarding particle in order to start an endoergic nuclear reaction.

See Figure 4.2 in Page 85 in Fundamentals of Nuclear Physics by Atam P. Arya for illustration a collision in the LAB coordinate system as well as in the CMCS.

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

A. BEFORE COLLISION

If a particle of mass m_x has velocity v_x in the LAB coordinate system, while the particle of mass M_X is at rest, the velocity v_c of the center-of-mass of the system is given by

$$(m_x + M_X)v_c = m_x v_x + M_X .0$$

$$v_c = \frac{m_x v_x}{m_x + M_X} \qquad (1)$$

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

Let us denote the velocities of mass m_x and M_X in the CMCS by v_x' and V_X' respectively, where

$$v_x' = v_x - v_c = v_x - \frac{m_x v_x}{m_x + M_X} = \frac{M_X}{m_x + M_X} v_x \qquad (2)$$

and

$$V_X' = 0 - v_c = -\frac{m_x v_x}{m_x + M_X} \qquad (3)$$

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

The kinetic energies, therefore, of the two particles before the collision in the CMCS are given by

$$K_x^{'} = \frac{1}{2} m_x v_x^{'\,2} = \frac{1}{2} m_x \left(\frac{M_X}{m_x + M_X} v_x \right)^2 = \left(\frac{M_X}{m_x + M_X} \right)^2 K_x \qquad (4)$$

and

$$K_X^{'} = \frac{1}{2} M_X V_X^{'\,2} = \frac{1}{2} M_X \left(\frac{-m_x v_x}{m_x + M_X} \right)^2 = \frac{m_x M_X}{\left(m_x + M_X\right)^2} K_x \qquad (5)$$

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

The total energy K_i' of the system before the collision in the CMCS is given by

$$K_i' = K_x' + K_X' = \left(\frac{M_X}{m_x + M_X}\right)^2 K_x + \frac{m_x M_X}{\left(m_x + M_X\right)^2} K_x = K_x \left(\frac{M_X}{m_x + M_X}\right) \qquad (6)$$

B. AFTER COLLISION

After the collision in CMCS, let v_y' and V_Y' be the velocities of the masses m_y and M_Y, respectively, and K_f' be the total kinetic energy of the system.

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

From the conservation of momentum, we have

$$m_y v_y = M_Y V_Y^{'} \qquad (7)$$

And the kinetic energies $K_y{}'$ and $K_Y{}'$ of m_y and M_Y in CMCS are

$$K_y^{'} = \frac{1}{2} m_y v_y^{'\,2} \qquad (8)$$

$$K_Y^{'} = \frac{1}{2} M_Y V_Y^{'\,2} = \frac{1}{2} M_Y \left(\frac{m_y v_y^{'}}{M_Y} \right)^2 = \frac{m_y}{M_Y} K_y^{'} \qquad (9)$$

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

Thus the total kinetic energy K_f' is given by

$$K_f' = K_y' + K_Y' = \frac{1}{2} m_y v_y'^2 + \frac{1}{2} M_Y V_Y'^2 \qquad (10)$$

But we must have

$$K_i' = K_f' - Q \qquad (11)$$

Substituting for K_i' from Eq. 6, we get

$$K_x \left(\frac{M_X}{m_x + M_X} \right) = K_f' - Q \Rightarrow K_f' = Q + K_x \left(\frac{M_X}{m_x + M_X} \right) = Q + K_x \left(1 - 1 + \frac{M_X}{m_x + M_X} \right) \qquad (12)$$

or

$$K_f' = Q + K_x \left(1 - \frac{m_x}{m_x + M_X} \right) \qquad (13)$$

NUCLEAR REACTIONS IN A CENTER-OF-MASS COORDINATE SYSTEM

It is interesting to compare this expression for $K_f{}'$ with K_f, given by

$$K_f = Q + K_x \qquad (14)$$

Using Eqs. 8 , 9 , 10 and 11, it can be shown that the kinetic energies $K_y{}'$ and $K_Y{}'$ after the collision , in the CMCS are given by

$$K_y^{'} = \frac{M_Y}{m_y + M_Y}\left[Q + \left(1 - \frac{m_x}{m_y + M_Y}\right)K_x\right] \qquad (15)$$

and

$$K_Y^{'} = \frac{m_y}{m_y + M_Y}\left[Q + \left(1 - \frac{m_x}{m_y + M_Y}\right)K_x\right] \qquad (16)$$

THRESHOLD ENERGY FOR AN ENDOERGIC REACTION IN A CENTER-OF-MASS COORDINATE SYSTEM

From the conservation of momentum and energy, we can calculate the threshold energy for endoergic reactions. In the previous section, we used the laboratory coordinate system and involved lengthy calculations for this aim. Let us consider the reaction in the center-of-mass coordinate system.

For a particle of mass m_x approaching another particle of mass M_X at rest with a velocity *v* in the LAB coordinate system, the energy in the CMCS from Eq. 6 is

$$K_i^{'} = \frac{1}{2} m_{red.} v^2 = \frac{1}{2}\left(\frac{m_x M_X}{\left(m_x + M_X\right)}\right) v^2 \qquad (17)$$

THRESHOLD ENERGY FOR AN ENDOERGIC REACTION IN A CENTER-OF-MASS COORDINATE SYSTEM

Thus the energy requirement in the CMCS for an endoergic reaction to take place will be

$$K_i^{'} \geq |Q| \qquad (18)$$

or from Eq. 17

$$\frac{1}{2}.\frac{m_x M_X}{m_x + M_X} v^2 \geq |Q| \quad \Rightarrow \quad \frac{1}{2} m_x v^2 \geq \frac{m_x + M_X}{M_X} |Q| \geq \left(1 + \frac{m_x}{M_X}\right) |Q| \qquad (19)$$

where $(m_x v^2/2) = K_x$ is the kinetic energy of the particle x in the LAB coordinate system; therefore

$$K_x \geq \left(1 + \frac{m_x}{M_X}\right)|Q| \quad hence \quad Threshold\ energy = \left(K_x\right)_{\min} = \left(1 + \frac{m_x}{M_X}\right)|Q| \qquad (20)$$

REFERENCES

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2. Fundamentals of Nuclear Physics. Atam. P. Arya