Advanced Quantum Mechanics II

PEN425

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Week 2

Inner Product

Normalization

Linear Operators

Outer Product

Matrix Representations of Operators

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Inner Product of a bra & ket

$$(\langle\beta|).(|\alpha\rangle) \overset{\text{dfn}}{=} \langle\beta|\alpha\rangle \qquad (\in C \text{ in general})$$

Property 1 : $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$

$$\langle\beta|\alpha\rangle \xrightarrow{\text{analog}} \mathbf{B.A}$$

$$\langle\alpha|\beta\rangle \xrightarrow{\text{analog}} \mathbf{A.B}$$

$$\langle\alpha|\alpha\rangle \in R; \langle\alpha|\alpha\rangle = \langle\alpha|\alpha\rangle^*$$

Property 2 : $\langle\alpha|\alpha\rangle \geq 0;\ \ if\ \langle\alpha|\alpha\rangle = 0$, then $|\alpha\rangle = 0$ (null vector).

Defn : Orthogonality of kets

$\langle\alpha|\beta\rangle = 0 \ \rightarrow |\alpha\rangle$ & $|\beta\rangle$ are orthogonal

Defn : Normalized ket

$$|\alpha\rangle \rightarrow |\tilde{\alpha}\rangle \equiv \frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}|\alpha\rangle$$

Then,

$$\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1$$

$\sqrt{\langle\alpha|\alpha\rangle}$: Norm of $|\alpha\rangle$ $\xrightarrow{\text{analog}}$ $|\boldsymbol{V}| = \sqrt{\boldsymbol{V}.\boldsymbol{V}} \equiv \sqrt{\boldsymbol{V}^2}$

Physical (Q) States $\leftrightarrow$ Normalized Kets

- <u>Operators</u>

X, Y; $X = Y$ if $X|\alpha\rangle = Y|\alpha\rangle$ (for arbitrary $|\alpha\rangle$)
$X = 0$ if $X|\alpha\rangle = 0$

- <u>Addition of Operators</u>
  - <u>Commutative</u> $X + Y = Y + X$
  - <u>Associative</u> $X + (Y + Z) = (X + Y) + Z$

Linear Operators

- $X(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1X|\alpha\rangle + c_2X|\beta\rangle$
- $X|\alpha\rangle \leftrightarrow \langle\alpha|X$ $\quad$ *are not dual correspondents*
- $X|\alpha\rangle \leftrightarrow \langle\alpha|X^{+}$ $\quad$ *are DC* $\;(X^{+}$ *is Hermitian adjoint of* $X)$
- If $X^{+} = X$; Hermitian Operators

- <u>Multiplication of Operators</u>
  - <u>Non-Commutative</u> $\quad XY \neq YX$
  - <u>Associative</u> $\quad X(YZ) = (XY)Z$
  - $(XY)^{+} = Y^{+}X^{+}$

Dfn: $Y|\alpha\rangle \equiv |\beta\rangle \leftrightarrow \langle\beta| \equiv \langle\alpha|Y^{+}$

$$XY|\alpha\rangle = X(Y|\alpha\rangle) = X|\beta\rangle \leftrightarrow$$
$$\langle\beta|X^{+} = \langle\alpha|Y^{+}X^{+} = \langle\alpha|(XY)^{+}$$

- <u>Outer Product</u>

$$(|\beta\rangle).(\langle\alpha|) = |\beta\rangle\langle\alpha|$$ → To be regarded as an operator?

- <u>Associativity of ‘Multiplication’ in general</u> (kets, bras, operators)
  Associative axiom of Multiplications

1. As an illustration, consider

$$\underbrace{[(|\beta\rangle).(\langle\alpha|)]}_{?}\underbrace{|\gamma\rangle}_{\text{Ket}} = \underbrace{|\beta\rangle}_{\text{Ket}}\underbrace{[\langle\alpha|\gamma\rangle]}_{\text{Number}}$$

- $|\beta\rangle\langle\alpha|$ is an operator, rotates the ‘ket’ to the direction of $|\beta\rangle$

- $X \overset{\text{dfn}}{=} |\beta\rangle\langle\alpha|$

  $X^{+} = |\alpha\rangle\langle\beta|$

$$X|\gamma\rangle \leftrightarrow \langle\gamma|X^{+}$$

$$|\beta\rangle\underbrace{\langle\alpha|\gamma\rangle}_{c} \leftrightarrow \langle\beta|c^{*} = \langle\beta|\langle\gamma|\alpha\rangle = \langle\gamma|\alpha\rangle\langle\beta| = \langle\gamma|(|\alpha\rangle\langle\beta|) = \langle\gamma|X^{+}$$

2. As an illustration (Ass. Axiom)

Consider, $\underbrace{(\langle\beta|).(X|\alpha\rangle) = (\langle\beta|X).|\alpha\rangle}$

Notation: $\langle\beta|X|\alpha\rangle$

$$\underbrace{X|\alpha\rangle}_{|\gamma\rangle} \leftrightarrow \underbrace{\langle\alpha|X^{+}}_{\langle\gamma|} \Rightarrow$$

$$\langle\beta|\gamma\rangle = \langle\gamma|\beta\rangle^{*} = [\langle\alpha|X^{+}|\beta\rangle]^{*}$$

$$\langle\beta|X|\alpha\rangle = \langle\alpha|X^{+}|\beta\rangle^{*}$$

↓

X: Hermitian

Matrix Representations of Operators

- Theorem: For a Hermitian operator *A,*
  - The eigenvalues are real
  - The eigenvectors corresponding to different eigenvalues are orthogonal

Proof:

$$A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$$

. $\langle\alpha_k|A|\alpha_i\rangle = \alpha_i\langle\alpha_k|\alpha_i\rangle$

$$\langle\alpha_k|A = \alpha_k{}^*\langle\alpha_k| \qquad ; A^+ = A$$

.. $\langle\alpha_k|A|\alpha_i\rangle = \alpha_k{}^*\langle\alpha_k|\alpha_i\rangle$

Subtract (. - ..); $0 = (\alpha_i - \alpha_k{}^*)\langle\alpha_k|\alpha_i\rangle$

- Take $i = k$,

  $$0 = (\alpha_i - \alpha_i{}^*)\langle\alpha_i|\alpha_i\rangle$$

  Hence $\langle\alpha_i|\alpha_i\rangle \neq 0$

  Then, $\alpha_i = \alpha_i{}^*$

- Take $i \neq k$,

  $$0 \neq \alpha_i - \alpha_k{}^*$$

  $$\langle\alpha_k|\alpha_i\rangle = 0$$

- Normalize eigenvectors $|\alpha_i\rangle$

$$\langle\alpha_k|\alpha_i\rangle = \delta_{ik}$$