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MÜHENDİSLİK FAKÜLTESİ

Advanced Quantum Mechanics II

PEN425

Dr. H.Ozgur Cildiroglu

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Week 3

Completeness of Eigenvectors

Matrix Representations

Spectral Decomposition

Measurement

Probability and Expectation Value

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Completeness of Eigenvectors

$$|\alpha\rangle = \sum_{i=1}^{n} c_i \, |\alpha_i\rangle$$

$$\langle \alpha_k | \alpha \rangle = \sum_i c_i \, \underbrace{\langle \alpha_k | \alpha_i \rangle}_{\delta_{ik}}$$

$$\begin{aligned} |\alpha\rangle &= \sum_i \langle \alpha_i | \alpha \rangle \, |\alpha_i\rangle \\ &= \sum_i |\alpha_i\rangle\langle \alpha_i | \alpha \rangle \\ &= (\sum_i |\alpha_i\rangle\langle \alpha_i |)|\alpha\rangle \end{aligned}$$

$$I = \sum_i |\alpha_i\rangle\langle \alpha_i|$$

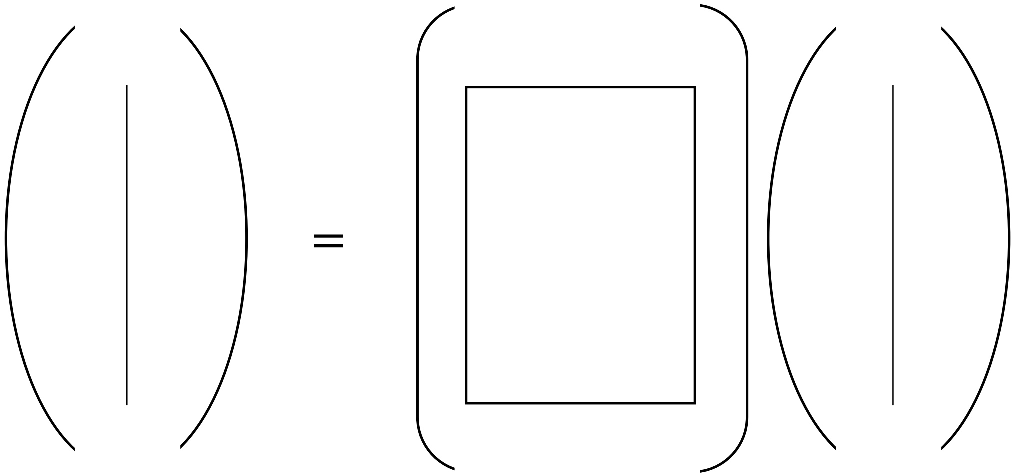
Matrix Representations

- $X = (\sum_i |\alpha_i\rangle\langle\alpha_i|)X(\sum_k |\alpha_k\rangle\langle\alpha_k|)$
  $= \sum_{i,k} |\alpha_i\rangle(\underbrace{\langle\alpha_i|X|\alpha_k\rangle}_{\text{Matrix representation of } X \text{ in the basis } \{|\alpha_i\rangle\}})\langle\alpha_k|$

$$|\beta\rangle = X|\alpha\rangle$$

$$\langle\alpha_i|\beta\rangle = \langle\alpha_i|X\underset{\substack{\uparrow \\ I = \sum_k |\alpha_k\rangle\langle\alpha_k|}}{\,}|\alpha\rangle$$

$$= \sum_k \langle\alpha_i|X|\alpha_k\rangle\langle\alpha_k|\alpha\rangle$$



- $|\beta\rangle = X|\alpha\rangle$

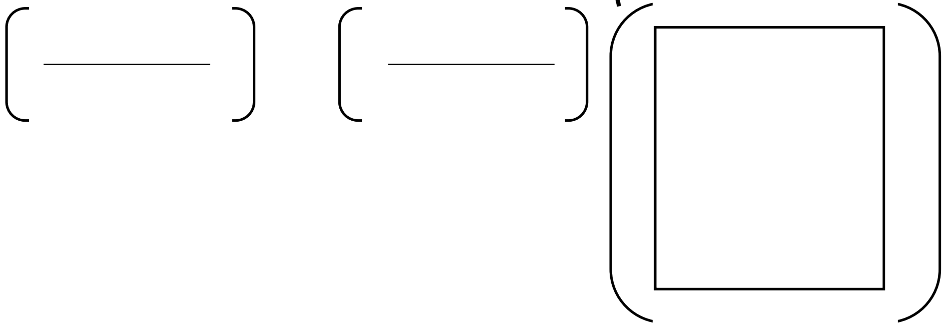
  $A_{ij}\ , B_{ij}$

  $(\mathrm{AB})_{ij} = \sum_k A_{ik}B_{kj}$

  $\langle\beta|\alpha_i\rangle = \langle\alpha|\ X|\alpha_i\rangle$

$$I = \sum_j |\alpha_j\rangle\langle\alpha_j|$$

$$\langle\beta|\alpha_i\rangle = \sum_j \langle\alpha|\alpha_j\rangle\langle\alpha_j\,|\,X|\alpha_i\rangle$$



$|\beta\rangle\langle\alpha|$ ←→ $\langle\alpha|\beta\rangle$

Outer Product ←→ Inner Product

(Operator) (C Number)

Matrix Representations in its own eigenvector basis

- $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$

$$\langle\alpha_k|A|\alpha_i\rangle = \alpha_i\underbrace{\langle\alpha_k|\alpha_i\rangle}_{\delta_{ik}}$$

$$= \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \ddots \end{pmatrix}$$

$$A = \Big(\sum_i |\alpha_i\rangle\langle\alpha_i|\Big)A\Big(\sum_k |\alpha_k\rangle\langle\alpha_k|\Big)$$

$$A = \sum_{i,k} |\alpha_i\rangle(\underbrace{\langle\alpha_i|A|\alpha_k\rangle)}_{\alpha_k\delta_{ik}}\langle\alpha_k|$$

$$= \sum_i |\alpha_i\rangle\alpha_i\langle\alpha_i|$$

$$= \sum_i \alpha_i\underbrace{|\alpha_i\rangle\langle\alpha_i|}_{\Lambda_i}$$

$$A = \sum_i \alpha_i\Lambda_i$$

Illustration

- $S_z \to |+\rangle_z, |-\rangle_z$

  $(+\frac{\hbar}{2})$ $(-\frac{\hbar}{2})$

- $I = \underbrace{|+\rangle\langle+|}_{\Lambda_+} + \underbrace{|-\rangle\langle-|}_{\Lambda_-}$

- $S_z = \frac{\hbar}{2}\Lambda_+ + (-\frac{\hbar}{2})\Lambda_-$

  $= \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$

- Matrix Representation in the $\{|\pm\rangle\}$ basis,

  $\langle\pm|\pm\rangle = 1, \langle\mp|\pm\rangle = 0$

$$|+\rangle \to \begin{pmatrix} \langle+|+\rangle \\ \langle-|+\rangle \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|-\rangle \to \begin{pmatrix} \langle+|-\rangle \\ \langle-|-\rangle \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- $I = |+\rangle\langle+| + |-\rangle\langle-|$

$$\langle+| \ldots |+\rangle = 1$$
$$\langle+| \ldots |-\rangle = 0$$
$$\langle-| \ldots |+\rangle = 0$$
$$\langle-| \ldots |-\rangle = 1$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

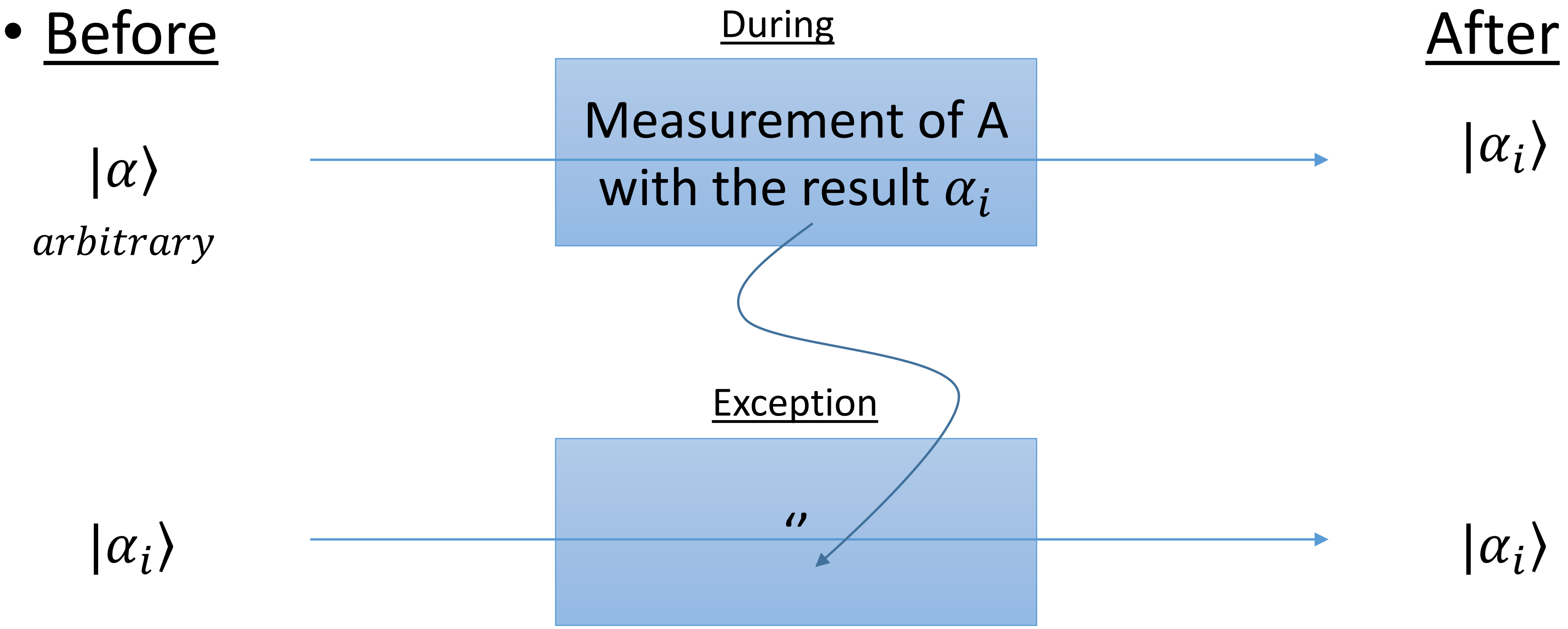
$$S_z = \begin{pmatrix} \langle+|S_z|+\rangle & \langle+|S_z|-\rangle \\ \langle-|S_z|+\rangle & \langle-|S_z|-\rangle \end{pmatrix}$$

$$\langle-|S_z|-\rangle = \frac{\hbar}{2}\langle-|(|+\rangle\langle+| - |-\rangle\langle-|)|-\rangle$$

$$\hbar|+\rangle\langle-| \equiv S_+ \rightarrow \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\hbar|-\rangle\langle+| \equiv S_- \rightarrow \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Measurement



- Before

During

After

$|\alpha\rangle$

$arbitrary$

Measurement of A with the result α_i

$|\alpha_i\rangle$

Exception

$|\alpha_i\rangle$

"

$|\alpha_i\rangle$

Probability

- Prob= $|\langle \alpha_i|\alpha\rangle|^2 \geq 0$
- $\sum |\langle \alpha_i|\alpha\rangle|^2 = 1$
- $\sum \langle \alpha|\alpha_i\rangle\langle \alpha_i|\alpha\rangle = \langle \alpha|\alpha\rangle = 1$

Expectation Value of an Operator

- $\langle A\rangle = \langle \alpha|A|\alpha\rangle$
- *Averaged Measured Value*

$$\langle \alpha| \quad A \quad |\alpha\rangle$$

$$I = \sum_i |\alpha_i\rangle\langle \alpha_i| \qquad I = \sum_k |\alpha_k\rangle\langle \alpha_k|$$

$$\langle A\rangle = \sum_{i,k} \langle \alpha|\alpha_i\rangle\langle \alpha_i|A|\alpha_k\rangle\langle \alpha_k|\alpha\rangle = \sum_i \alpha_i \langle \alpha|\alpha_i\rangle\langle \alpha_i|\alpha\rangle$$

$$\langle A\rangle_\alpha = \sum_i \alpha_i |\langle \alpha_i|\alpha\rangle|^2$$

Measured Values

Prob. of obtaining α_i