Advanced Quantum Mechanics II

PEN425

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Week 4

Compatible Operators

Measurement of Compatible Operators

Incompatible Observables

Heisenberg Uncertainty Relations

Schwarz Inequality and Heisenberg Uncertainty Principle

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Compatible Operators

- $\exists\ A, B \ \rightarrow [A, B] = 0 \quad Compatible \longrightarrow \{\boldsymbol{S}^2, S_z\}$

  $[A, B] \neq 0 \quad In - Compatible \longrightarrow \{S_z, S_x\}$

  $A \downarrow \{|\alpha_i\rangle\}$ $\quad B \downarrow \{|b_i\rangle\}$

  $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$

  } How these basis are related?

- Consider non-degenerate case
  - One $\alpha_i \longleftrightarrow$ One $|\alpha_i\rangle$; *non-degenerate*
  - One $\alpha_i \longleftrightarrow$ Many $|\alpha_i\rangle$; *degenerate*

Theorem:

$[A, B] = 0$

Assume, eigenvectors of A are non-degenerate,

$A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle \rightarrow \langle\alpha_i|A|\alpha_k\rangle = \alpha_i \delta_{ik}$, diagonal

Claim: $\langle\alpha_i|B|\alpha_k\rangle$ is diagonal.

Proof:

$\langle\alpha_i|[A, B]|\alpha_k\rangle = 0$

$(\alpha_i - \alpha_k)\langle\alpha_i|B|\alpha_k\rangle = 0$

$i)$ $\alpha_i = \alpha_k$; $\langle\alpha_i|B|\alpha_i\rangle \neq 0$

$ii)$ $\alpha_i \neq \alpha_k$; $\langle\alpha_i|B|\alpha_k\rangle = 0$

$$B = \Big(\sum_i |\alpha_i\rangle\langle\alpha_i|\Big) B \Big(\sum_k |\alpha_k\rangle\langle\alpha_k|\Big)$$

$$= \sum_{i,k} |\alpha_i\rangle\big(\langle\alpha_i|B|\alpha_k\rangle\big)\langle\alpha_k| \quad \longleftarrow \quad \langle\alpha_i|B|\alpha_k\rangle = \delta_{ik}\langle\alpha_i|B|\alpha_i\rangle$$

$$= \sum_i |\alpha_i\rangle\langle\alpha_i|B|\alpha_i\rangle\langle\alpha_i|$$

$$B|\alpha_k\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|B|\alpha_i\rangle\langle\alpha_i|\alpha_k\rangle \quad \longleftarrow \quad \langle\alpha_i\alpha_k\rangle = \delta_{ik}$$

$$= |\alpha_k\rangle\underbrace{\big(\langle\alpha_k|B|\alpha_k\rangle\big)}_{\textit{eigenvalues of } B \textit{ in } \{|\alpha_k\rangle\} \textit{ basis}}$$

- $|\alpha_i\rangle$: Simultaneous eigenkets of *A, B*

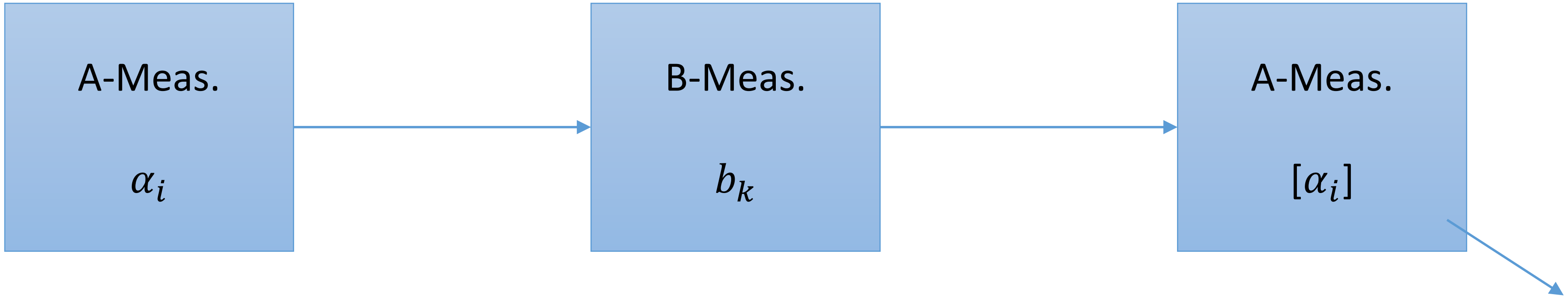
$$|\alpha_i, b_i\rangle \quad \begin{cases} A|\alpha_i, b_k\rangle = \alpha_i|\alpha_i, b_k\rangle \\ B|\alpha_i, b_k\rangle = b_k|\alpha_i, b_k\rangle \end{cases}$$

- This statements hold for the degenerate case as well

$$A|\alpha_i{}^{(k)}\rangle = \alpha_i|\alpha_i{}^{(k)}\rangle\,; \quad \begin{array}{l} i = 1, \ldots, N \;\rightarrow Dim. of\ full\ ket\ space \\ k = 1, \ldots, N \rightarrow Dim. of\ degenerate\ sub\ space \end{array}$$

Measurement of Compatible Operators

$$[A, B] = 0$$



$get\ back\ the\ same\ \alpha_i$

$Note\ that: 2nd\ measurement\ (B) does\ not\ destroy\ the\ previous\ info$

Incompatible Observables

- $[A, B] \neq 0$

<u>Claim</u>: Do NOT have a complete set of Simultaneous eigenstates.

<u>Proof</u>: Assume contrary;

B/ $A|\alpha_i, b_k\rangle$=$\alpha_i|\alpha_i, b_k\rangle$

A/ $B|\alpha_i, b_k\rangle$=$b_k|\alpha_i, b_k\rangle$

$[B, A]|\alpha_i, b_k\rangle = 0 \rightarrow [A, B] = 0$ → CONTRADICTION!

- <u>Exception</u>

  - $\{\boldsymbol{L}^2, L_z\}$: Compatible $\{l, m\}$: $complete\ orthonormal$
  - $\{L_x, L_y\}$: Incompatible $\rightarrow \nexists\ a\ complete\ common\ eigenset$

- Special case $l = 0$ subspace (1d)

  - $L_x|0,0\rangle$=*0*
  - $L_y|0,0\rangle$=*0*

  $\Big\}$ $|0,0\rangle$ is simultaneous eigenstate of L_x and L_x with 0 eigenvalues

Heisenberg Uncertainty Relations

- $[A, B] \neq 0$

Define: $\Delta A \equiv A - \langle A \rangle I$

$\langle (\Delta A)^2 \rangle$: Dispersion of A-Operator

$$\langle (\Delta A)^2 \rangle = \langle A^2 - 2\langle A \rangle A + \langle A \rangle^2 I \rangle = \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2 I$$
$$= \langle A^2 \rangle - \langle A \rangle^2$$

Note that!

- If $|\psi\rangle$ is an eigenstate of A ; $|\psi\rangle = |\alpha_i\rangle$

$\langle\alpha_i|A|\alpha_i\rangle = \alpha_i\langle\alpha_i|\alpha_i\rangle = \alpha_i$
$A^2|\alpha_i\rangle = {\alpha_i}^2|\alpha_i\rangle$
$\langle\alpha_i|A^2|\alpha_i\rangle = {\alpha_i}^2$

$\langle(\Delta A)^2\rangle$=${\alpha_i}^2$ -$(\alpha_i)^2$ =0

Heisenberg Uncertainty Theorem

- $[A, B] \neq 0$

$$\boxed{\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2}$$

$$\left.\begin{array}{l} A = x \\ B = p_x \end{array}\right\} \; [x, p_x] = i\hbar I$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \frac{1}{4}|\langle[x, p_x]\rangle|^2 = \frac{1}{4}|i\hbar|^2 = \frac{\hbar^2}{4}$$

$$\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}}$$

Recitation:

- Prove Schwarz inequality.
- Prove Heisenberg Uncertainty Principle by using Schwarz inequality.