Advanced Quantum Mechanics II

PEN425

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Week 5

Change of Basis

Matrix Representations of Transformation Operators

Transformation of Coordinates

Transformation of Operators

Trace of an Operator

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Change of Basis

- $[A, B] \neq 0$

$$\begin{tikzcd}
& {[A,B]} \arrow[dl] \arrow[dr] & \\
\underset{old}{\{|\alpha_i\rangle\}} \arrow[rr, leftrightarrow, "U"] & & \underset{new}{\{|b_i\rangle\}}
\end{tikzcd}$$

Theorem: There exists a Unitary Operator U, s.t. $|b_i\rangle = U|\alpha_i\rangle$

Proof: $U = \sum_k |b_k\rangle\langle a_k|$

$$U|a_i\rangle = \sum_k |b_k\rangle\langle a_k|a_i\rangle = \sum_k |b_k\rangle\delta_{ki} = |b_i\rangle$$

$$Demonstration\ of\ Unitarity:$$

$$U^{+}U = UU^{+} = I$$

$$U = \sum_{k} |b_k\rangle\langle a_k|$$

$$U^{+} = \sum_{k} |a_k\rangle\langle b_k|$$

$$\underbrace{U^{+}U = \sum_{k,l} |a_l\rangle\langle b_l|b_k\rangle\langle a_k| = \sum_{k,l} \delta_{lk}|a_l\rangle\langle a_k| = \sum_{k}|a_k\rangle\langle a_k| = I}_{Completeness\ of\ a-basis}$$

Matrix Representations of Transformation Operators

- Old Basis: $\langle \alpha_i | U | \alpha_j \rangle$

$$U = \sum_k |b_k\rangle\langle a_k|$$

$$\langle \alpha_i | U | \alpha_j \rangle = \sum_k \langle \alpha_i | b_k \rangle \langle a_k | \alpha_j \rangle = \sum_k \langle \alpha_i | b_k \rangle \, \delta_{kj} = \langle \alpha_i | b_j \rangle$$

$$\langle \alpha_i | U | \alpha_j \rangle = \begin{pmatrix} U_{11} & U_{12} & \cdots \\ U_{21} & U_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

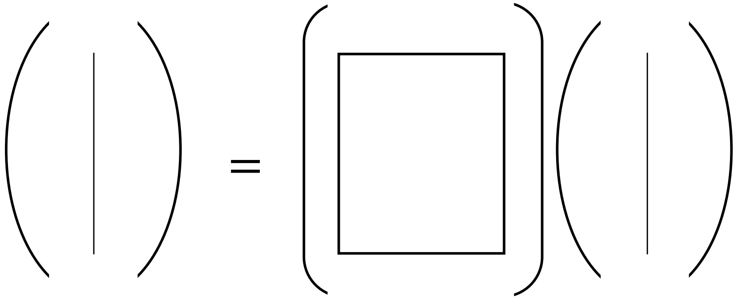
Transformation of Coordinates

$$|a\rangle = \sum_i |a_i\rangle\langle a_i|a\rangle$$

$$\underbrace{\langle b_j|a\rangle}_{New\ Coords.} = \sum_i \underbrace{\langle b_j|a_i\rangle}\underbrace{\langle a_i|a\rangle}_{Old\ Coords.}$$

$$\boxed{\langle \alpha_j|U^+|\alpha_i\rangle}$$

$$\langle \alpha_i|U|\alpha_j\rangle = \langle \alpha_i|b_j\rangle$$



$$(New\ Coords) = U^+(Old\ Coords)$$

Transformation of Operators under U

$$\langle b_i| \quad X \quad |b_j\rangle$$

$$I = \sum_l |\alpha_l\rangle\langle\alpha_l| \qquad I = \sum_k |\alpha_k\rangle\langle\alpha_k|$$

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \underbrace{\langle b_i|\alpha_l\rangle}_{\langle\alpha_i|U^+|\alpha_l\rangle} \underbrace{\langle\alpha_l|X|\alpha_k\rangle} \underbrace{\langle\alpha_k|b_j\rangle}_{\langle\alpha_k|U|\alpha_j\rangle}$$

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \langle\alpha_i|U^+|\alpha_l\rangle\langle\alpha_l|X|\alpha_k\rangle\langle\alpha_k|U|\alpha_j\rangle$$

$$\boxed{X \rightarrow U^+XU = X'}$$

Transformation of Operators under U

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \langle b_i|\alpha_l\rangle\langle\alpha_l|X|\alpha_k\rangle\langle\alpha_k|b_j\rangle$$

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \langle\alpha_i|U^+|\alpha_l\rangle\langle\alpha_l|X|\alpha_k\rangle\langle\alpha_k|U|\alpha_j\rangle$$

$$\boxed{X \rightarrow U^+XU = X'}$$

Old $\qquad\qquad$ New

Trace of an Operator

$$Tr(X) \equiv \sum_k \langle \alpha_k | X | \alpha_k \rangle$$

$Lemma: Trace\ is\ independent\ of\ representation$

$$Tr(X) = \sum_k \langle \alpha_k | \ X \ | \alpha_k \rangle$$

$$I = \sum_l |\alpha_l\rangle\langle\alpha_l|$$

$$Tr(X) = \sum_{k,l,m} \langle \alpha_k | b_l \rangle \langle b_l | X | b_m \rangle \langle b_m | \alpha_k \rangle$$

$$Tr(X) = \sum_{k,l,m} \langle \alpha_k | b_l \rangle \langle b_l | X | b_m \rangle \boxed{\langle b_m | \alpha_k \rangle}$$

$$Tr(X) = \sum_{l,m} \{ \sum_{k} \langle b_m | \alpha_k \rangle \langle \alpha_k | b_l \rangle \} \langle b_l | X | b_m \rangle$$

$$= \sum_{l,m} \{ \sum_{k} \langle b_m | \alpha_k \rangle \langle \alpha_k | b_l \rangle \} \langle b_l | X | b_m \rangle$$

$$= \sum_{l,m} \langle b_m | b_l \rangle \langle b_l | X | b_m \rangle = \sum_{l,m} \delta_{ml} \langle b_l | X | b_m \rangle$$

$$\boxed{Tr(X) = \sum_{m} \langle b_m | X | b_m \rangle}$$

X & Y any operators,

- $Tr(XY) = Tr(YX)$
- $Tr(U^{+}XU) = Tr(X)$
- $Tr\big(|\alpha_k\rangle\langle\alpha_l|\big) = \delta_{kl}$
- $Tr\big(|b_k\rangle\langle\alpha_l|\big) = \langle\alpha_l|b_k\rangle$

Recitation:

- Prove all these four Trace properties.