Advanced Quantum Mechanics II

PEN425

Dr. H.Ozgur Cildiroglu

Advanced Quantum Mechanics II

PEN425

Week 6

Unitary Equivalence

Continious Spectra

Position Operator

Momentum Operator

Ankara University | Physics Engineering Department

Dr. H. Ozgur Cildiroglu

Unitary Equivalance

$|b_i\rangle = U|\alpha_i\rangle$

$\{|\alpha_i\rangle\} \leftrightarrow \{|b_i\rangle\}$ *given* $(U \text{ known})$

$A \qquad U \qquad B$

Can determine U

$$A \rightarrow A' \equiv UAU^{-1}$$

Unitary equivalent observables

- $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$; $(U|\alpha_i\rangle = |b_i\rangle)$
- $U(A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle)$
- $(UAU^{-1})(U|\alpha_i\rangle) = \alpha_i(U|\alpha_i\rangle)$
- $(UAU^{-1})|b_i\rangle = \alpha_i|b_i\rangle$

Compare against

- $B|b_i\rangle = b_i|b_i\rangle$

A & UAU^{-1}

B & UAU^{-1}

Have common eigenvectors
(Can be diagonalized together)

$$\langle b_i|B|b_j\rangle = b_i\delta_{ij}$$

$$\langle b_i|UAU^{-1}|b_j\rangle = a_i\delta_{ij}$$

Continious Spectra

- $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$; $i = 1,2,\ldots,N$
- Instead $\xi^{op}|\xi\rangle = \xi\,|\xi\rangle$

$\xi^{op} \downarrow x, p_x, \ldots$ $\qquad \xi \downarrow$ *continious*

- Orthonormality: $\langle\alpha_i|\alpha_j\rangle = \delta_{ij} \longrightarrow \langle\xi|\xi'\rangle = \delta(\xi - \xi')$

$$I = \sum_i |\alpha_i\rangle\langle\alpha_i| \longrightarrow \int d\xi\,|\xi\rangle\langle\xi| = I$$

- $$|\alpha\rangle = \sum_i \langle\alpha_i|\alpha\rangle|\alpha_i\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|\alpha\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|\,\alpha\rangle$$

$$|\alpha\rangle = \int d\xi\,|\xi\rangle\underbrace{\langle\xi|\alpha\rangle}_{Coordinate}$$

- $$|\langle\xi|\alpha\rangle|^2 \longrightarrow \int d\xi\,|\langle\xi|\alpha\rangle|^2 = 1$$

- $$\langle\beta|\alpha\rangle = \int d\xi\,\underbrace{\langle\xi|\beta\rangle^*}_{\beta^*(\xi)}\underbrace{\langle\xi|\alpha\rangle}_{\alpha(\xi)}$$

Position Operator

- $X|x\rangle = x|x\rangle$
- Postulate: $\{|x\rangle\}$

  Orthonormal ; $\langle x|x'\rangle = \delta(x - x')$

  Complete ; $I = \int dx|x\rangle\langle x|$

$$|\alpha\rangle = \left(\int dx|x\rangle\langle x|\right)|\alpha\rangle = \int dx|x\rangle\underbrace{\langle x|\alpha\rangle}_{\alpha(x)}$$

$$\int dx|\langle x|\alpha\rangle|^2 = 1 \quad \leftrightarrow \langle x|\alpha\rangle = 1$$

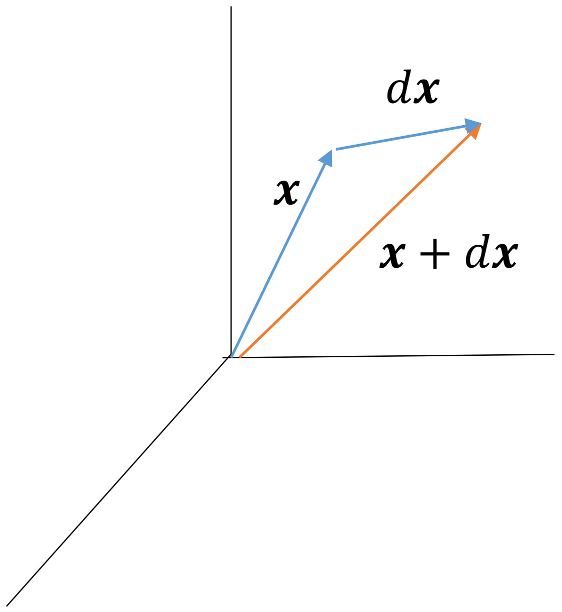
Momentum Operator

- Translation in space

$$\tau(d\boldsymbol{x})$$

$$\tau_{d\boldsymbol{x}}$$

$$\tau(d\boldsymbol{x})|\boldsymbol{x}\rangle \overset{dfn}{\equiv} |\boldsymbol{x} + d\boldsymbol{x}\rangle$$



Note that: $|\boldsymbol{x}\rangle$ is NOT eigenvector of τ

- $|\alpha\rangle$: Arbitrary state

$$|\alpha\rangle_{tr} \equiv \tau|\alpha\rangle \leftrightarrow \quad {}_{tr}\langle a| = \langle a|\tau^{+}$$

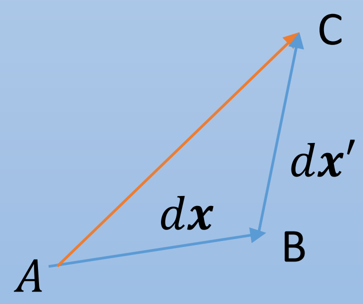
$i)$ Preservation of the normalisation

$${}_{tr}\langle a|\alpha\rangle_{tr} = \langle a|\alpha\rangle = I$$

$$\langle a|\tau^{+}\tau|\alpha\rangle = \langle a|\alpha\rangle$$

$$\tau^{+}\tau = I$$

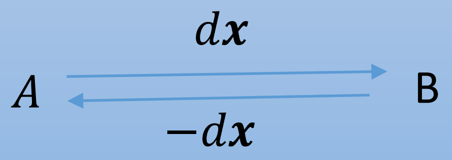
$ii)$ Group Property



$$\tau(d\boldsymbol{x})\tau(d\boldsymbol{x}') = \tau(d\boldsymbol{x} + d\boldsymbol{x}')$$

$iii)$

$$\tau(-d\boldsymbol{x}) = \tau^{-1}(d\boldsymbol{x})$$



$iv)$

$$\lim_{d\boldsymbol{x}\to 0} \tau(d\boldsymbol{x}) = I$$

Construction of $\tau(d\boldsymbol{x})$

$$\tau(d\boldsymbol{x}) \overset{dfn}{\equiv} I - id\boldsymbol{x}.\underset{\uparrow}{\boldsymbol{K}}$$

(Hermitian)

Check Unitarity: $\tau^{+}\tau = (I + id\boldsymbol{x}.\boldsymbol{K})(I - id\boldsymbol{x}.\boldsymbol{K}) = I$

By definition, $\boldsymbol{X}/\ \ \tau(d\boldsymbol{x})|\boldsymbol{x}\rangle = |\boldsymbol{x} + d\boldsymbol{x}\rangle$

$$\boldsymbol{X}\tau(d\boldsymbol{x})|\boldsymbol{x}\rangle = \boldsymbol{X}|\boldsymbol{x} + d\boldsymbol{x}\rangle = (\boldsymbol{x} + d\boldsymbol{x})|\boldsymbol{x} + d\boldsymbol{x}\rangle$$

$$\tau/\ \ \boldsymbol{X}|\boldsymbol{x}\rangle = \mathbf{x}|\boldsymbol{x}\rangle$$

$$\tau\boldsymbol{X}|\boldsymbol{x}\rangle = \boldsymbol{x}\tau|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x} + d\boldsymbol{x}\rangle$$

$$[\boldsymbol{X},\tau] = d\boldsymbol{x}|\boldsymbol{x} + d\boldsymbol{x}\rangle$$

- $|\boldsymbol{x} + d\boldsymbol{x}\rangle = |\boldsymbol{x}\rangle + \underbrace{d\boldsymbol{x}.\boldsymbol{\nabla}|\boldsymbol{x}\rangle}_{dx_j \nabla_j}$

- $[\boldsymbol{X},\tau]|\boldsymbol{x}\rangle \approx dx_i\left(|\boldsymbol{x}\rangle + dx_j\nabla_j|\boldsymbol{x}\rangle\right)$

$$\approx dx_i|\boldsymbol{x}\rangle + dx_i dx_j \nabla_j |\boldsymbol{x}\rangle \ldots$$

$$\approx dx_i|\boldsymbol{x}\rangle + O(d\boldsymbol{x}^2)$$

- $[X_i,\tau] \approx dx_i$

$\downarrow$

$I - id\boldsymbol{x}.\boldsymbol{K}$

$$-idx_j[X_i,K_j] = dx_i = \delta_{ij}dx_j$$

$$[X_i,K_j] = i\delta_{ij}$$

$$\updownarrow$$

$$K_i \equiv \frac{P_i}{\hbar}$$

$$\tau(d\boldsymbol{x}) = I - \frac{i}{\hbar}\boldsymbol{P}.d\boldsymbol{x}$$

$$[X_i, P_j] = i\hbar\delta_{ij}$$