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MÜHENDİSLİK FAKÜLTESİ

Advanced Quantum Mechanics II

PEN425

Dr. H.Ozgur Cildiroglu

Advanced Quantum Mechanics II

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Week 1

Introduction

State

Stern – Gerlach Experiment

Dirac Notation

Ket and Bra Spaces

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Books

I. J.J. Sakurai, Modern Quantum Mechanics (2nd Ed.)
II. E. Merzbacher, QM (3rd Ed.)

NRQM

III. J. Bjorken, S. Drell, RQM (v.1)
IV. J.J. Sakurai, Advanced QM
V. W. Greiner, D.A. Bromley, RQM

RQM

NRQM

- Hydrogen Atom – Fine Structure (Addition of Angular Momenta)
- Quantum Mechanical Pictures – Time Evolution Problem
    - Schrödinger Picture
    - Heisenberg Picture
    - Dirac Picture
- Time Dependent Problems

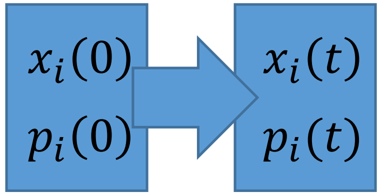
- Time Dependent Perturbation Theory.
    - Constant Potentials
    - Harmonic Potentials
- Atoms in Classical Radiation Field
- Gauge Problems
- Spontaneous and Simulated emmissions

RQM

- Lorentz Transformations
- First attempt on Relativistic QM. Klein Gordon Eqns.
- Dirac Eqn.
- Non-Relativistic Limits
  - Free, in pure magnetic field (g=2)
  - Coulomb field (Fine Structures of Hydrogen)
- Classical Limits

State

- Classical Physics: $\{x_i, p_i\}$
- $\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}(m\boldsymbol{v}) = m\frac{d^2\boldsymbol{x}}{dt^2}$
- One can ‘predict’ $\boldsymbol{x}(t), \boldsymbol{p}(t)$
- Need ‘2 initial conditions’ for ∀i



$$\begin{matrix} x_i(0) \\ p_i(0) \end{matrix} \Rightarrow \begin{matrix} x_i(t) \\ p_i(t) \end{matrix}$$

- ‘Causality’
- ‘Deterministic’
- ‘Physics’ is Phenomenological Science

⇩

Measurement
(Experiment & Observations)

- Quantization of Charge: $Q = ne$

Measurement in ‘microworld’ is order dependent.

⇩

We need ‘order dependent’ mathematical entities to describe ‘dynamical variables’

⇩

Operators/Matrices

Stern-Gerlach Experiment (1921-22)

- 1900: Planck
- 1905: Einstein
- 1926:

Schrödinger
Heisenberg
Dirac
Born

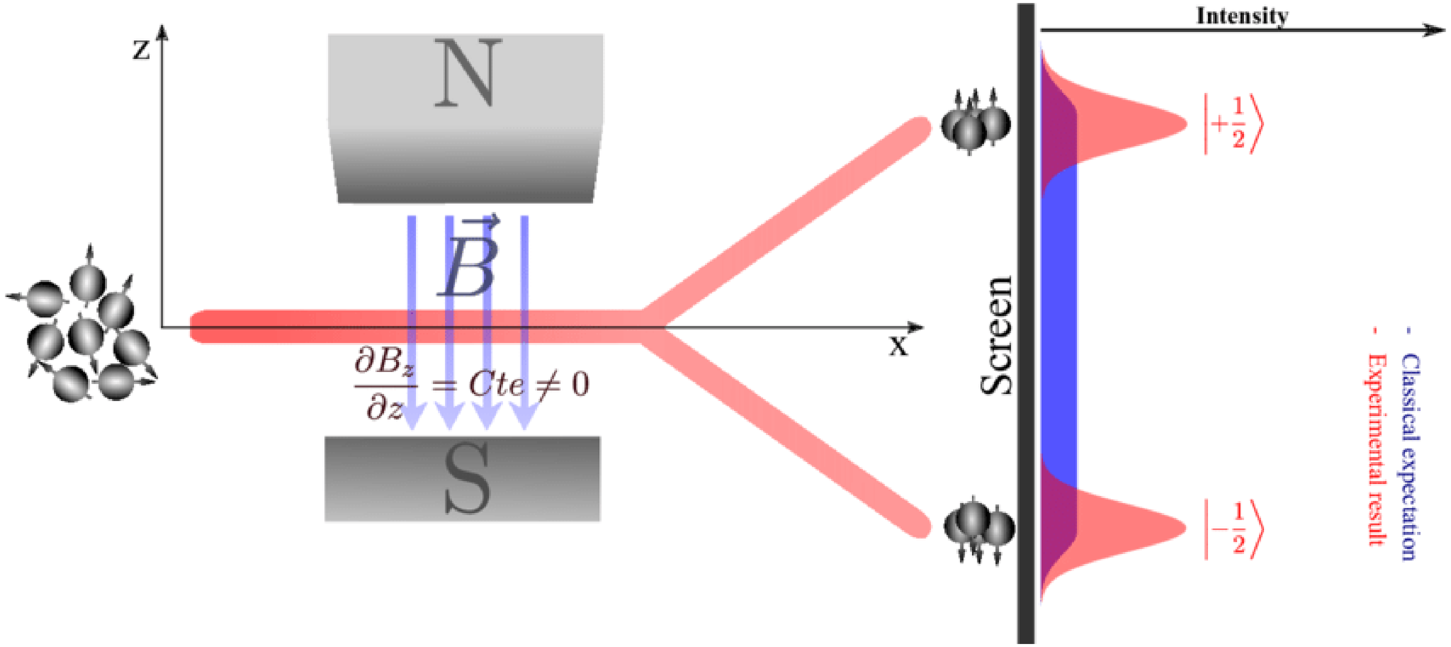


Figure: https://www.researchgate.net/figure/Sketch-of-Stern-Gerlach-experiment-A-beam-of-silver-atoms-is-sent-through-an_fig2_334586888

Classical

State: $\{x_i, p_i\}$

Dynamical Variables (c-numbers)

Quantum

Vectors in a Linear Complex Vector Space*

Hermitian (Linear) Operators

*Dimensions of Vector Space : Number of Linearly independent Basis Vectors

- Orthogonal
- Normalized
- Complete

Orthonormal

*Number of possible results of an experiment

- Hilbert Space: Infinite Dim.
- Complex Vector Space = $\{\underset{|\alpha\rangle}{\underline{\text{State Vectors}}}\ ;\ \underset{A}{\underline{\text{Observables}}}\}$

Exp: Spin $\frac{1}{2}$

$$S_z|\pm\rangle_z = \pm\frac{\hbar}{2}|\pm\rangle_z$$
$$S_x|\pm\rangle_x = \pm\frac{\hbar}{2}|\pm\rangle_x$$

$$A(|\alpha\rangle)=A|\alpha\rangle \neq (const)|\alpha\rangle$$

$$\left.\begin{array}{l}\text{If } A|\alpha\rangle = (const)|\alpha\rangle \\ |\alpha\rangle\text{: } eigenstate(ket)\ of\ A \\ \{\alpha_i; i = 1, \ldots, n\} \\ c_i \in C\end{array}\right\} \quad A|\alpha_i\rangle=\alpha_i|\alpha_i\rangle$$

$$\underset{eigenvalues}{\alpha_i} \qquad \underset{eigenkets}{|\alpha_i\rangle}$$

$\Rightarrow$ Arbitrary State

$$|\alpha\rangle = \sum_{i=1}^{n} c_i\,|\alpha_i\rangle$$

Ket Space	Dual Correspondance	Bra Space
$\|\alpha\rangle$	↔	$\langle\alpha\|$
$\{\|\alpha_i\rangle\}$	↔	$\{\langle\alpha_i\|\}$
$\|\alpha\rangle + \|\beta\rangle$	↔	$\langle\alpha\| + \langle\beta\|$
$\mathrm{c}\|\alpha\rangle$	↔	$\mathrm{c}^*\langle\alpha\|$
$c_1\|\alpha\rangle + c_2\|\beta\rangle$	↔	$c_1{}^*\langle\alpha\| + c_2{}^*\langle\beta\|$

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Week 2

Inner Product

Normalization

Linear Operators

Outer Product

Matrix Representations of Operators

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Inner Product of a bra & ket

$$(\langle\beta|).(|\alpha\rangle) \overset{\text{dfn}}{=} \langle\beta|\alpha\rangle \qquad (\in C \text{ in general})$$

Property 1 : $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$

$$\langle\beta|\alpha\rangle \xrightarrow{\text{analog}} \mathbf{B.A}$$

$$\langle\alpha|\beta\rangle \xrightarrow{\text{analog}} \mathbf{A.B}$$

$$\langle\alpha|\alpha\rangle \in R; \langle\alpha|\alpha\rangle = \langle\alpha|\alpha\rangle^*$$

Property 2 : $\langle\alpha|\alpha\rangle \geq 0;\ \ if\ \langle\alpha|\alpha\rangle = 0$, then $|\alpha\rangle = 0$ (null vector).

Defn : Orthogonality of kets

$$\langle\alpha|\beta\rangle = 0 \ \rightarrow |\alpha\rangle \text{ \& } |\beta\rangle \text{ are orthogonal}$$

<u>Defn</u> : Normalized ket

$$|\alpha\rangle \;\rightarrow |\tilde{\alpha}\rangle \equiv \frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}|\alpha\rangle$$

Then,

$$\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1$$

$\sqrt{\langle\alpha|\alpha\rangle}$: Norm of $|\alpha\rangle$ $\xrightarrow{\text{analog}}$ $|\boldsymbol{V}| = \sqrt{\boldsymbol{V}.\boldsymbol{V}} \equiv \sqrt{\boldsymbol{V}^2}$

Physical (Q) States $\leftrightarrow$ Normalized Kets

- Operators

$X, Y;$ $\quad X = Y$ if $X|\alpha\rangle = Y|\alpha\rangle$ (for arbitrary $|\alpha\rangle$)
$X = 0$ if $X|\alpha\rangle = 0$

- Addition of Operators
  - Commutative $\quad X + Y = Y + X$
  - Associative $\quad X + (Y + Z) = (X + Y) + Z$

Linear Operators

- $X(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1X|\alpha\rangle + c_2X|\beta\rangle$
- $X|\alpha\rangle \leftrightarrow \langle\alpha|X \qquad are\ not\ dual\ correspondents$
- $X|\alpha\rangle \leftrightarrow \langle\alpha|X^+ \qquad are\ DC \quad (X^+\ is\ Hermitian\ adjoint\ of\ X)$
- If $X^+ = X$; Hermitian Operators

- <u>Multiplication of Operators</u>
  - <u>Non-Commutative</u> $\qquad XY \neq YX$
  - <u>Associative</u> $\qquad X(YZ) = (XY)Z$
  - $(XY)^+ = Y^+X^+$

Dfn: $Y|\alpha\rangle \equiv |\beta\rangle \leftrightarrow \langle\beta| \equiv \langle\alpha|Y^+$

$$XY|\alpha\rangle = X(Y|\alpha\rangle) = X|\beta\rangle \leftrightarrow$$
$$\langle\beta|X^+ = \langle\alpha|Y^+X^+ = \langle\alpha|(XY)^+$$

- <u>Outer Product</u>

$$(|\beta\rangle).(\langle\alpha|) = |\beta\rangle\langle\alpha|$$

→ To be regarded as an operator?

- <u>Associativity of ‘Multiplication’ in general</u> (kets, bras, operators)
  Associative axiom of Multiplications

1. As an illustration, consider

$$\underbrace{[(|\beta\rangle).(\langle\alpha|)]}_{?}\underbrace{|\gamma\rangle}_{\text{Ket}} = \underbrace{|\beta\rangle}_{\text{Ket}}\underbrace{[\langle\alpha|\gamma\rangle]}_{\text{Number}}$$

- $|\beta\rangle\langle\alpha|$ is an operator, rotates the ‘ket’ to the direction of $|\beta\rangle$

- $X \overset{\text{dfn}}{=} |\beta\rangle\langle\alpha|$
  $X^+ = |\alpha\rangle\langle\beta|$

$$X|\gamma\rangle \leftrightarrow \langle\gamma|X^+$$
$$|\beta\rangle\underbrace{\langle\alpha|\gamma\rangle}_{c} \leftrightarrow \langle\beta|c^* = \langle\beta|\langle\gamma|\alpha\rangle = \langle\gamma|\alpha\rangle\langle\beta| = \langle\gamma|(|\alpha\rangle\langle\beta|) = \langle\gamma|X^+$$

2. As an illustration (Ass. Axiom)

Consider, $\underbrace{(\langle\beta|).(X|\alpha\rangle) = (\langle\beta|X).|\alpha\rangle}_{\langle\beta|X|\alpha\rangle}$

Notation: $\langle\beta|X|\alpha\rangle$

$$\underbrace{X|\alpha\rangle}_{|\gamma\rangle} \leftrightarrow \underbrace{\langle\alpha|X^+}_{\langle\gamma|} \quad \Longrightarrow \quad \begin{gathered}\langle\beta|\gamma\rangle = \langle\gamma|\beta\rangle^* = [\langle\alpha|X^+|\beta\rangle]^* \\ \langle\beta|X|\alpha\rangle = \langle\alpha|X^+|\beta\rangle^*\end{gathered}$$

$\downarrow$

X: Hermitian

Matrix Representations of Operators

- Theorem: For a Hermitian operator *A,*
  - The eigenvalues are real
  - The eigenvectors corresponding to different eigenvalues are orthogonal

Proof:

$$A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$$

. $\langle\alpha_k|A|\alpha_i\rangle = \alpha_i\langle\alpha_k|\alpha_i\rangle$

$\langle\alpha_k|A = \alpha_k{}^*\langle\alpha_k|$ $\quad ; A^+ = A$

.. $\langle\alpha_k|A|\alpha_i\rangle = \alpha_k{}^*\langle\alpha_k|\alpha_i\rangle$

Subtract (. - ..); $\quad 0 = (\alpha_i - \alpha_k{}^*)\langle\alpha_k|\alpha_i\rangle$

- Take $i = k$,

  $$0 = (\alpha_i - \alpha_i{}^*)\langle\alpha_i|\alpha_i\rangle$$

  Hence $\langle\alpha_i|\alpha_i\rangle \neq 0$

  Then, $\quad \alpha_i = \alpha_i{}^*$

- Take $i \neq k$,

  $$0 \neq \alpha_i - \alpha_k{}^*$$

  $$\langle\alpha_k|\alpha_i\rangle = 0$$

- Normalize eigenvectors $|\alpha_i\rangle$

$$\boxed{\langle\alpha_k|\alpha_i\rangle = \delta_{ik}}$$

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Week 3

Completeness of Eigenvectors

Matrix Representations

Spectral Decomposition

Measurement

Probability and Expectation Value

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Completeness of Eigenvectors

$$|\alpha\rangle = \sum_{i=1}^{n} c_i \, |\alpha_i\rangle$$

$$\langle\alpha_k|\alpha\rangle = \sum_i c_i \, \underbrace{\langle\alpha_k|\alpha_i\rangle}_{\delta_{ik}}$$

$$\begin{aligned} |\alpha\rangle &= \sum_i \langle\alpha_i|\alpha\rangle \, |\alpha_i\rangle \\ &= \sum_i |\alpha_i\rangle\langle\alpha_i|\alpha\rangle \\ &= (\sum_i |\alpha_i\rangle\langle\alpha_i|)|\alpha\rangle \end{aligned}$$

$$I = \sum_i |\alpha_i\rangle\langle\alpha_i|$$

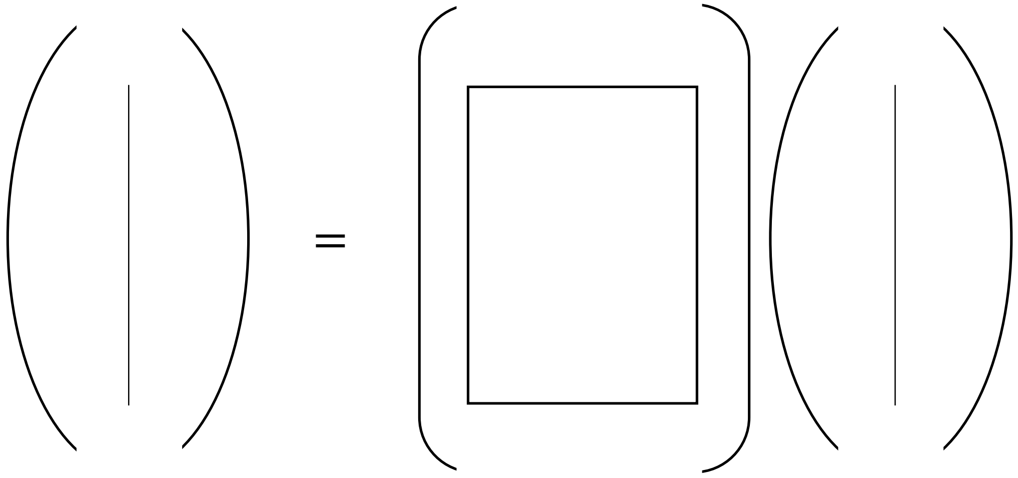
Matrix Representations

- $X = (\sum_i |\alpha_i\rangle\langle\alpha_i|)X(\sum_k |\alpha_k\rangle\langle\alpha_k|)$
  $= \sum_{i,k} |\alpha_i\rangle(\underbrace{\langle\alpha_i|X|\alpha_k\rangle}_{\text{Matrix representation of } X \text{ in the basis } \{|\alpha_i\rangle\}})\langle\alpha_k|$

$$|\beta\rangle = X|\alpha\rangle$$

$$\langle\alpha_i|\beta\rangle = \langle\alpha_i|X \underset{\substack{\uparrow \\ I = \sum_k |\alpha_k\rangle\langle\alpha_k|}}{\,} |\alpha\rangle$$

$$=\sum_k \langle\alpha_i|X|\alpha_k\rangle\langle\alpha_k|\,\alpha\rangle$$



- $|\beta\rangle = X|\alpha\rangle$

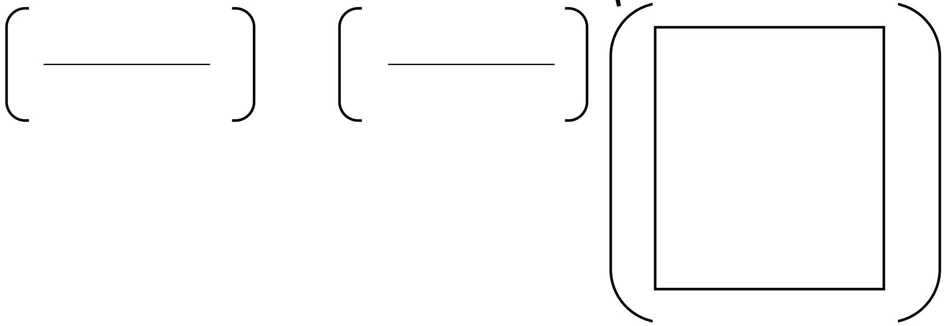
  $A_{ij}\ , B_{ij}$

  $(\mathrm{AB})_{ij} = \sum_k A_{ik} B_{kj}$

  $\langle\beta|\alpha_i\rangle = \langle\alpha|\ X|\alpha_i\rangle$

$$I = \sum_j |\alpha_j\rangle\langle\alpha_j|$$

$$\langle\beta|\alpha_i\rangle = \sum_j \langle\alpha|\alpha_j\rangle\langle\alpha_j\,|\,X|\alpha_i\rangle$$



$|\beta\rangle\langle\alpha|$

Outer Product ↔ Inner Product

(Operator)

$\langle\alpha|\beta\rangle$

Inner Product

(C Number)

Matrix Representations in its own eigenvector basis

- $A|\alpha_i\rangle=\alpha_i|\alpha_i\rangle$

$$\langle\alpha_k|A|\alpha_i\rangle=\alpha_i\underbrace{\langle\alpha_k|\alpha_i\rangle}_{\delta_{ik}}$$

$$=\begin{pmatrix}\alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \ddots\end{pmatrix}$$

$$A=(\sum_i|\alpha_i\rangle\langle\alpha_i|)A(\sum_k|\alpha_k\rangle\langle\alpha_k|)$$

$$A=\sum_{i,k}|\alpha_i\rangle(\underbrace{\langle\alpha_i|A|\alpha_k\rangle)}_{\alpha_k\delta_{ik}})\langle\alpha_k|$$

$$=\sum_i|\alpha_i\rangle\alpha_i\langle\alpha_i|$$

$$=\sum_i\alpha_i\underbrace{|\alpha_i\rangle\langle\alpha_i|}_{\Lambda_i}$$

$$A=\sum_i\alpha_i\Lambda_i$$

Illustration

- $S_z \rightarrow \underset{(+\frac{\hbar}{2})}{|+\rangle_z}, \underset{(-\frac{\hbar}{2})}{|-\rangle_z}$

- $I = \underbrace{|+\rangle\langle+|}_{\Lambda_+} + \underbrace{|-\rangle\langle-|}_{\Lambda_-}$

- $S_z = \frac{\hbar}{2}\Lambda_+ + (-\frac{\hbar}{2})\Lambda_-$

$$= \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$$

- Matrix Representation in the $\{|\pm\rangle\}$ basis, $\langle\pm|\pm\rangle = 1, \langle\mp|\pm\rangle = 0$

$$|+\rangle \rightarrow \begin{pmatrix} \langle+|+\rangle \\ \langle-|+\rangle \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|-\rangle \rightarrow \begin{pmatrix} \langle+|-\rangle \\ \langle-|-\rangle \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- $I = |+\rangle\langle+| + |-\rangle\langle-|$

$$\langle+|\ldots|+\rangle = 1$$
$$\langle+|\ldots|-\rangle = 0$$
$$\langle-|\ldots|+\rangle = 0$$
$$\langle-|\ldots|-\rangle = 1$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

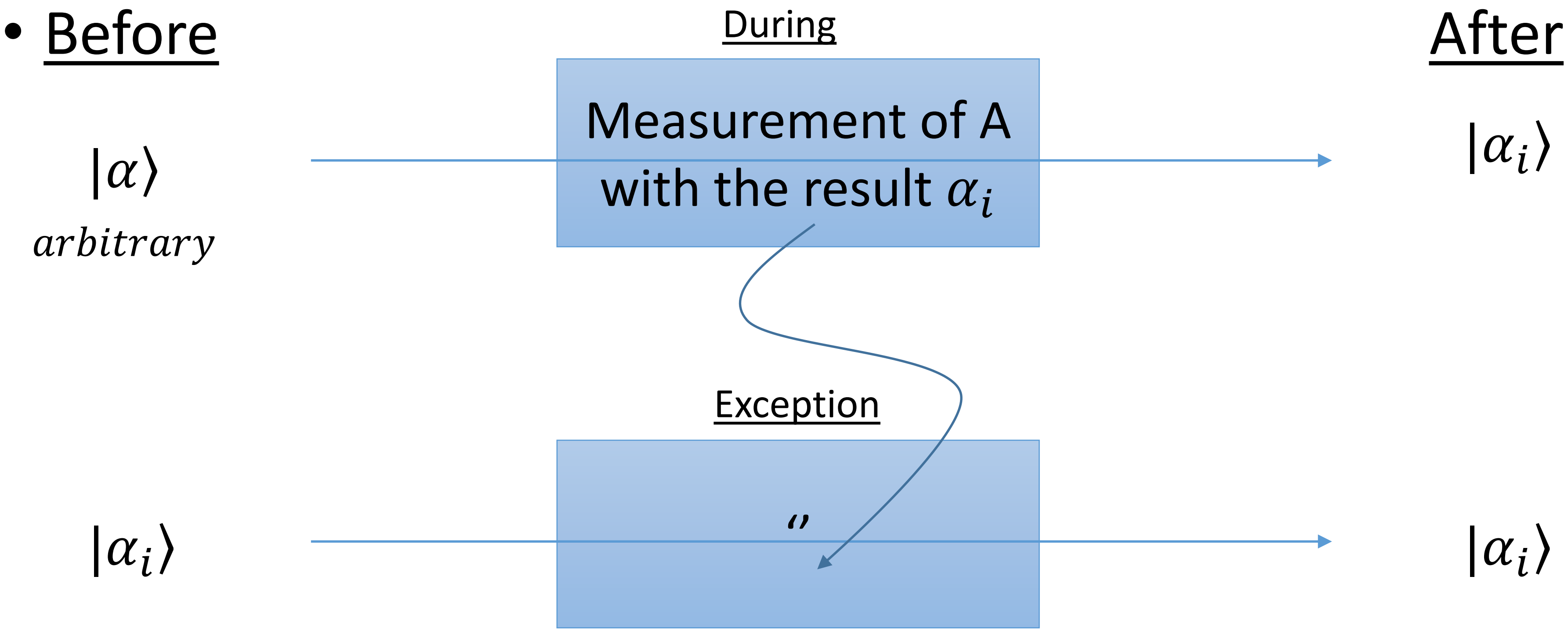
$$S_z = \begin{pmatrix} \langle+|S_z|+\rangle & \langle+|S_z|-\rangle \\ \langle-|S_z|+\rangle & \langle-|S_z|-\rangle \end{pmatrix}$$

$$\langle-|S_z|-\rangle=\frac{\hbar}{2}\langle-|(|+\rangle\langle+| - |-\rangle\langle-|)|-\rangle$$

$$\hbar|+\rangle\langle-| \equiv S_+ \rightarrow \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\hbar|-\rangle\langle+| \equiv S_- \rightarrow \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Measurement



- Before

During

After

$|\alpha\rangle$

$arbitrary$

Measurement of A with the result α_i

$|\alpha_i\rangle$

Exception

$|\alpha_i\rangle$

“

$|\alpha_i\rangle$

Probability

- Prob= $|\langle\alpha_i|\alpha\rangle|^2 \geq 0$
- $\sum|\langle\alpha_i|\alpha\rangle|^2 = 1$
- $\sum\langle\alpha|\alpha_i\rangle\langle\alpha_i|\alpha\rangle = \langle\alpha|\alpha\rangle = 1$

Expectation Value of an Operator

- $\langle A\rangle = \langle\alpha|A|\alpha\rangle$
- *Averaged Measured Value*

$$\langle\alpha| \underset{\uparrow}{\ } \; A \; \underset{\uparrow}{\ } |\alpha\rangle$$

$$I = \sum_i |\alpha_i\rangle\langle\alpha_i| \qquad I = \sum_k |\alpha_k\rangle\langle\alpha_k|$$

$$\langle A\rangle = \sum_{i,k}\langle\alpha|\alpha_i\rangle\langle\alpha_i|A|\alpha_k\rangle\langle\alpha_k|\alpha\rangle = \sum_i \alpha_i\langle\alpha|\alpha_i\rangle\langle\alpha_i|\alpha\rangle$$

$$\boxed{\langle A\rangle_\alpha = \sum_i \underset{\uparrow}{\alpha_i} \underset{\uparrow}{|\langle\alpha_i|\alpha\rangle|^2}}$$

Measured Values

Prob. of obtaining α_i

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Week 4

Compatible Operators

Measurement of Compatible Operators

Incompatible Observables

Heisenberg Uncertainty Relations

Schwarz Inequality and Heisenberg Uncertainty Principle

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Compatible Operators

- $\exists\ A, B \rightarrow [A, B] = 0 \quad Compatible \longrightarrow \{\boldsymbol{S}^2, S_z\}$

  $[A, B] \neq 0 \quad In - Compatible \longrightarrow \{S_z, S_x\}$

$$\left.\begin{array}{l} \{|\alpha_i\rangle\} \quad \{|b_i\rangle\} \\ A|\alpha_i\rangle=\alpha_i|\alpha_i\rangle \end{array}\right\}$$ How these basis are related?

- Consider non-degenerate case
  - One $\alpha_i \longleftrightarrow$ One $|\alpha_i\rangle$; *non-degenerate*
  - One $\alpha_i \longleftrightarrow$ Many $|\alpha_i\rangle$; $degenerate$

Theorem:

$[A, B] = 0$

Assume, eigenvectors of *A* are non-degenerate,

$A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle \rightarrow \langle\alpha_i|A|\alpha_k\rangle = \alpha_i\delta_{ik}$, diagonal

Claim: $\langle\alpha_i|B|\alpha_k\rangle$ is diagonal.

Proof:

$\langle\alpha_i|[A, B]|\alpha_k\rangle = 0$

$(\alpha_i - \alpha_k)\langle\alpha_i|B|\alpha_k\rangle = 0$

$i)$ $\alpha_i = \alpha_k$; $\langle\alpha_i|B|\alpha_i\rangle \neq 0$

$ii)$ $\alpha_i \neq \alpha_k$; $\langle\alpha_i|B|\alpha_k\rangle = 0$

$$B = (\sum_i |\alpha_i\rangle\langle\alpha_i|)B(\sum_k |\alpha_k\rangle\langle\alpha_k|)$$

$$= \sum_{i,k} |\alpha_i\rangle(\langle\alpha_i|B|\alpha_k\rangle)\langle\alpha_k| \longleftarrow \langle\alpha_i|B|\alpha_k\rangle = \delta_{ik}\langle\alpha_i|B|\alpha_i\rangle$$

$$= \sum_i |\alpha_i\rangle\langle\alpha_i|B|\alpha_i\rangle\langle\alpha_i|$$

$$B|\alpha_k\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|B|\alpha_i\rangle\langle\alpha_i|\alpha_k\rangle \longleftarrow \langle\alpha_i\alpha_k\rangle = \delta_{ik}$$

$$= |\alpha_k\rangle\underbrace{(\langle\alpha_k|B|\alpha_k\rangle)}_{eigenvalues\ of\ B\ in\ \{|\alpha_k\rangle\}\ basis}$$

- $|\alpha_i\rangle$: Simultaneous eigenkets of *A, B*

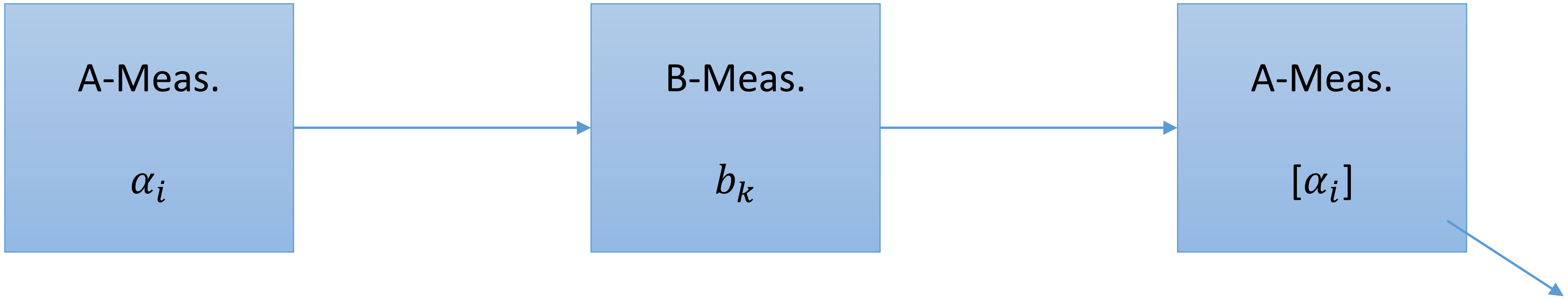
$$|\alpha_i, b_i\rangle \quad \begin{matrix} A|\alpha_i, b_k\rangle = \alpha_i|\alpha_i, b_k\rangle \\ B|\alpha_i, b_k\rangle = b_k|\alpha_i, b_k\rangle \end{matrix}$$

- This statements hold for the degenerate case as well

$$A|\alpha_i{}^{(k)}\rangle = \alpha_i|\alpha_i{}^{(k)}\rangle\ ; \quad \begin{matrix} i = 1, \dots, N \ \rightarrow Dim. of\ full\ ket\ space \\ k = 1, \dots, N \rightarrow Dim. of\ degenerate\ sub\ space \end{matrix}$$

Measurement of Compatible Operators

$$[A, B] = 0$$



A-Meas.

α_i

B-Meas.

b_k

A-Meas.

$[\alpha_i]$

get back the same α_i

Note that: *2nd measurement* (B)*does not destroy the previous info*

Incompatible Observables

- $[A, B] \neq 0$

Claim: Do NOT have a complete set of Simultaneous eigenstates.

Proof: Assume contrary;

B/ $A|\alpha_i, b_k\rangle$=$\alpha_i|\alpha_i, b_k\rangle$

A/ $B|\alpha_i, b_k\rangle$=$b_k|\alpha_i, b_k\rangle$

$[B, A]|\alpha_i, b_k\rangle = 0 \rightarrow [A, B] = 0$ → CONTRADICTION!

- <u>Exception</u>

  - $\{\boldsymbol{L}^2, L_z\}$: Compatible $\{l, m\}$: $complete\ orthonormal$
  - $\{L_x, L_y\}$: Incompatible $\rightarrow \nexists\ a\ complete\ common\ eigenset$

- Special case $l = 0$ subspace (1d)

$$\left.\begin{array}{l} \bullet\ L_x|0{,}0\rangle = 0 \\ \bullet\ L_y|0{,}0\rangle = 0 \end{array}\right\}$$

$|0{,}0\rangle$ is simultaneous eigenstate of L_x and L_x with 0 eigenvalues

Heisenberg Uncertainty Relations

- $[A, B] \neq 0$

Define: $\Delta A \equiv A - \langle A \rangle I$

$\langle (\Delta A)^2 \rangle$: Dispersion of A-Operator

$$\langle (\Delta A)^2 \rangle = \langle A^2 - 2\langle A \rangle A + \langle A \rangle^2 I \rangle = \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2 I$$

$$= \langle A^2 \rangle - \langle A \rangle^2$$

Note that!

- If $|\psi\rangle$ is an eigenstate of A ; $|\psi\rangle = |\alpha_i\rangle$

$$\left\langle \alpha_i|A|\alpha_i \right\rangle = \alpha_i \left\langle \alpha_i|\alpha_i \right\rangle = \alpha_i$$

$$A^2|\alpha_i\rangle = {\alpha_i}^2|\alpha_i\rangle$$

$$\left\langle \alpha_i|A^2|\alpha_i \right\rangle = {\alpha_i}^2$$

$\langle(\Delta A)^2\rangle$=${\alpha_i}^2$ -$(\alpha_i)^2$ =0

Heisenberg Uncertainty Theorem

- $[A, B] \neq 0$

$$\boxed{\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2}$$

$$\left.\begin{array}{l} A = x \\ B = p_x \end{array}\right\} \; [x, p_x] = i\hbar I$$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_x)^2 \rangle \geq \frac{1}{4} |\langle [x, p_x] \rangle|^2 = \frac{1}{4} |i\hbar|^2 = \frac{\hbar^2}{4}$$

$$\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}}$$

Recitation:

- Prove Schwarz inequality.
- Prove Heisenberg Uncertainty Principle by using Schwarz inequality.

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Week 5

Change of Basis

Matrix Representations of Transformation Operators

Transformation of Coordinates

Transformation of Operators

Trace of an Operator

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Change of Basis

$$\begin{tikzcd}
& \bullet\ {[A,B] \neq 0} \arrow[dl] \arrow[dr] & \\
{\underset{old}{\{|\alpha_i\rangle\}}} \arrow[rr, leftrightarrow, "U"] & & {\underset{new}{\{|b_i\rangle\}}}
\end{tikzcd}$$

Theorem: There exists a Unitary Operator U, s.t. $|b_i\rangle$=$U|\alpha_i\rangle$

Proof: $U = \sum_k |b_k\rangle\langle a_k|$

$$U|a_i\rangle = \sum_k |b_k\rangle\langle a_k|a_i\rangle = \sum_k |b_k\rangle\delta_{ki} = |b_i\rangle$$

Demonstration of Unitarity:

$$U^+U = UU^+ = I$$

$$U = \sum_k |b_k\rangle\langle a_k|$$

$$U^+ = \sum_k |a_k\rangle\langle b_k|$$

$$\underbrace{U^+U = \sum_{k,l} |a_l\rangle\langle b_l|b_k\rangle\langle a_k| = \sum_{k,l} \delta_{lk}|a_l\rangle\langle a_k| = \sum_k |a_k\rangle\langle a_k| = I}_{Completeness\ of\ a - basis}$$

Matrix Representations of Transformation Operators

- Old Basis: $\langle \alpha_i | U | \alpha_j \rangle$

$$U = \sum_k |b_k\rangle\langle a_k|$$

$$\langle \alpha_i | U | \alpha_j \rangle = \sum_k \langle \alpha_i | b_k \rangle \langle a_k \, | \alpha_j \rangle = \sum_k \langle \alpha_i | b_k \rangle \, \delta_{kj} = \langle \alpha_i | b_j \rangle$$

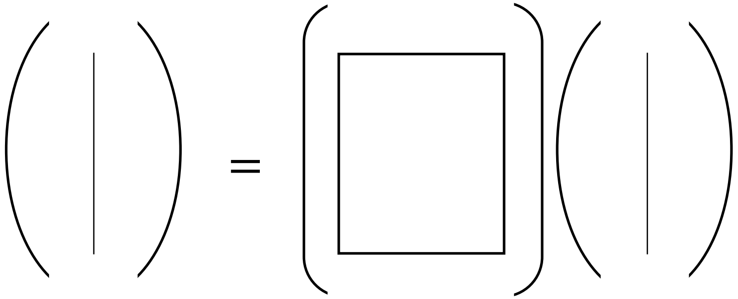
$$\langle \alpha_i | U | \alpha_j \rangle = \begin{pmatrix} U_{11} & U_{12} & \cdots & \\ U_{21} & U_{22} & \cdots & \\ \vdots & \vdots & & \end{pmatrix}$$

Transformation of Coordinates

$$|a\rangle = \sum_i |a_i\rangle\langle a_i|a\rangle$$

$$\underbrace{\langle b_j|a\rangle}_{New\ Coords.} = \sum_i \underbrace{\langle b_j|a_i\rangle}\underbrace{\langle a_i|a\rangle}_{Old\ Coords.}$$

$$\boxed{\langle \alpha_j|U^+|\alpha_i\rangle} \Leftarrow \langle \alpha_i|U|\alpha_j\rangle = \langle \alpha_i|b_j\rangle$$



$$\Rightarrow (New\ Coords) = U^+(Old\ Coords)$$

Transformation of Operators under *U*

$$\langle b_i| \underset{\uparrow}{} X \underset{\uparrow}{} |b_j\rangle$$

$$I = \sum_l |\alpha_l\rangle\langle\alpha_l| \qquad I = \sum_k |\alpha_k\rangle\langle\alpha_k|$$

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \underbrace{\langle b_i|\alpha_l\rangle}_{\substack{\downarrow \\ \langle\alpha_i|U^+|\alpha_l\rangle}} \underbrace{\langle\alpha_l|X|\alpha_k\rangle}_{} \underbrace{\langle\alpha_k|b_j\rangle}_{\substack{\downarrow \\ \langle\alpha_k|U|\alpha_j\rangle}}$$

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \langle\alpha_i|U^+|\alpha_l\rangle\langle\alpha_l|X|\alpha_k\rangle\langle\alpha_k|U|\alpha_j\rangle$$

$$\boxed{X \rightarrow U^+XU = X'}$$

Transformation of Operators under *U*

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \langle b_i|\alpha_l\rangle\langle\alpha_l|X|\alpha_k\rangle\langle\alpha_k|b_j\rangle$$

$$\langle b_i|X|b_j\rangle = \sum_{l,k} \langle\alpha_i|U^+|\alpha_l\rangle\langle\alpha_l|X|\alpha_k\rangle\langle\alpha_k|U|\alpha_j\rangle$$

$$\boxed{X \to U^+XU = X'}$$

$$\underset{Old}{\downarrow} \qquad\qquad\qquad \underset{New}{\downarrow}$$

Trace of an Operator

$$Tr(X) \equiv \sum_k \langle \alpha_k | X | \alpha_k \rangle$$

$Lemma: Trace\ is\ independent\ of\ representation$

$$Tr(X) = \sum_k \langle \alpha_k | \underset{\uparrow}{\ } X \underset{\uparrow}{\ } | \alpha_k \rangle$$

$$I = \sum_l |\alpha_l\rangle\langle\alpha_l|$$

$$Tr(X) = \sum_{k,l,m} \langle \alpha_k | b_l \rangle \langle b_l | X | b_m \rangle \langle b_m | \alpha_k \rangle$$

$$Tr(X) = \sum_{k,l,m} \langle \alpha_k | b_l \rangle \langle b_l | X | b_m \rangle \boxed{\langle b_m | \alpha_k \rangle}$$

$$Tr(X) = \sum_{l,m} \{ \sum_{k} \langle b_m | \alpha_k \rangle \langle \alpha_k | b_l \rangle \} \langle b_l | X | b_m \rangle$$

$$= \sum_{l,m} \{ \sum_{k} \langle b_m | \alpha_k \rangle \langle \alpha_k | b_l \rangle \} \langle b_l | X | b_m \rangle$$

$$= \sum_{l,m} \langle b_m | b_l \rangle \langle b_l | X | b_m \rangle = \sum_{l,m} \delta_{ml} \langle b_l | X | b_m \rangle$$

$$\boxed{Tr(X) = \sum_{m} \langle b_m | X | b_m \rangle}$$

X & Y any operators,

- $Tr(XY) = Tr(YX)$
- $Tr(U^{+}XU) = Tr(X)$
- $Tr\big(|\alpha_k\rangle\langle\alpha_l|\big) = \delta_{kl}$
- $Tr\big(|b_k\rangle\langle\alpha_l|\big) = \langle\alpha_l|b_k\rangle$

Recitation:

- Prove all these four Trace properties.

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Week 6

Unitary Equivalence

Continious Spectra

Position Operator

Momentum Operator

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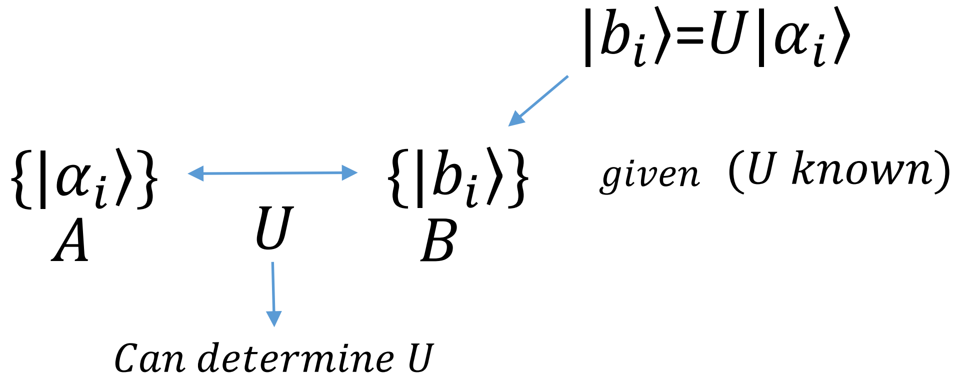
Unitary Equivalalance



$|b_i\rangle = U|\alpha_i\rangle$

$$\begin{tikzcd} \{|\alpha_i\rangle\} \arrow[r, leftrightarrow, "U"'] & \{|b_i\rangle\} \end{tikzcd}$$

A B

given $(U \text{ known})$

Can determine U

$$\underbrace{A \to A'}_{\textit{Unitary equivalent observables}} \equiv UAU^{-1}$$

- $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$; $(U|\alpha_i\rangle = |b_i\rangle)$
- $U(A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle)$
- $(UAU^{-1})(U|\alpha_i\rangle) = \alpha_i(U|\alpha_i\rangle)$
- $(UAU^{-1})|b_i\rangle = \alpha_i|b_i\rangle$

↕ Compare against

- $B|b_i\rangle = b_i|b_i\rangle$

$$A \,\&\, UAU^{-1}$$
$$B \,\&\, UAU^{-1}$$

Have common eigenvectors
(Can be diagonalized together)

$$\langle b_i|B|b_j\rangle = b_i\delta_{ij}$$
$$\langle b_i|UAU^{-1}|b_j\rangle = a_i\delta_{ij}$$

Continious Spectra

- $A|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$; $i = 1,2,\ldots,N$
- Instead $\underset{\substack{\downarrow \\ x,\, p_x, \ldots}}{\xi^{op}}|\xi\rangle = \underset{\substack{\downarrow \\ \mathit{continious}}}{\xi}\,|\xi\rangle$
- Orthonormality: $\langle\alpha_i|\alpha_j\rangle = \delta_{ij} \longrightarrow \langle\xi|\xi'\rangle = \delta(\xi - \xi')$

$$I = \sum_i |\alpha_i\rangle\langle\alpha_i| \longrightarrow \int d\xi |\xi\rangle\langle\xi| = I$$

- $$|\alpha\rangle = \sum_i \langle\alpha_i|\alpha\rangle|\alpha_i\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|\alpha\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|\,\alpha\rangle$$

  $$|\alpha\rangle = \int d\xi\,|\xi\rangle\underbrace{\langle\xi|\alpha\rangle}_{Coordinate}$$

- $$|\langle\xi|\alpha\rangle|^2 \longrightarrow \int d\xi|\langle\xi|\alpha\rangle|^2 = 1$$

- $$\langle\beta|\alpha\rangle = \int d\xi\underbrace{\langle\xi|\beta\rangle^*}_{\beta^*(\xi)}\underbrace{\langle\xi|\alpha\rangle}_{\alpha(\xi)}$$

Position Operator

- $X|x\rangle = x|x\rangle$

- Postulate: $\{|x\rangle\}$

Orthonormal ; $\langle x|x'\rangle = \delta(x - x')$

Complete ; $I = \int dx|x\rangle\langle x|$

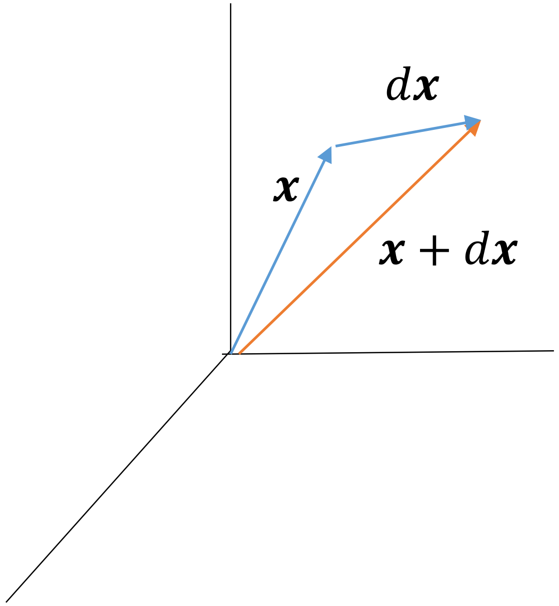
$$|\alpha\rangle = \left(\int dx|x\rangle\langle x|\right)|\alpha\rangle = \int dx|x\rangle\underbrace{\langle x|\alpha\rangle}_{\alpha(x)}$$

$$\int dx|\langle x|\alpha\rangle|^2 = 1 \quad \leftrightarrow \langle x|\alpha\rangle = 1$$

Momentum Operator

- Translation in space

$$\begin{tikzcd}
\tau(d\boldsymbol{x}) \arrow[d, leftrightarrow] \arrow[ddr] & \\
\tau_{d\boldsymbol{x}} & \\
& \tau(d\boldsymbol{x})|\boldsymbol{x}\rangle \overset{dfn}{\equiv} |\boldsymbol{x}+d\boldsymbol{x}\rangle
\end{tikzcd}$$



Note that: $|\boldsymbol{x}\rangle$ is NOT eigenvector of τ

- $|\alpha\rangle$: Arbitrary state

$$|\alpha\rangle_{tr} \equiv \tau|\alpha\rangle \leftrightarrow \quad {}_{tr}\langle a| = \langle a|\tau^{+}$$

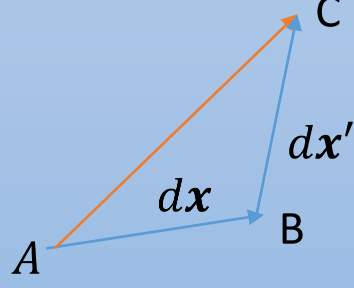
$i)$ Preservation of the normalisation

$$ {}_{tr}\langle a|\alpha\rangle_{tr} = \langle a|\alpha\rangle = I$$

$$\langle a|\tau^{+}\tau|\alpha\rangle = \langle a|\alpha\rangle$$

$$\tau^{+}\tau = I$$

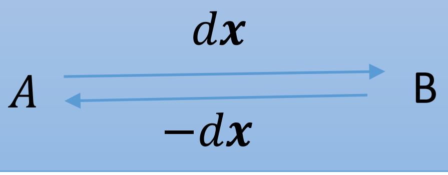
$ii)$ Group Property



$$\tau(d\boldsymbol{x})\tau(d\boldsymbol{x}') = \tau(d\boldsymbol{x} + d\boldsymbol{x}')$$

$iii)$

$$\tau(-d\boldsymbol{x}) = \tau^{-1}(d\boldsymbol{x})$$



$iv)$ $$\lim_{d\boldsymbol{x}\to 0} \tau(d\boldsymbol{x}) = I$$

Construction of $\tau(d\boldsymbol{x})$

$$\tau(d\boldsymbol{x}) \overset{dfn}{\equiv} I - id\boldsymbol{x}.\overset{\overset{Hermitian}{\uparrow}}{\boldsymbol{K}}$$

Check Unitarity: $\tau^{+}\tau = (I + id\boldsymbol{x}.\boldsymbol{K})(I - id\boldsymbol{x}.\boldsymbol{K}) = I$

By definition, $\boldsymbol{X}/\ \ \tau(d\boldsymbol{x})|\boldsymbol{x}\rangle = |\boldsymbol{x} + d\boldsymbol{x}\rangle$

$$\boldsymbol{X}\tau(d\boldsymbol{x})|\boldsymbol{x}\rangle = \boldsymbol{X}|\boldsymbol{x} + d\boldsymbol{x}\rangle = (\boldsymbol{x} + d\boldsymbol{x})|\boldsymbol{x} + d\boldsymbol{x}\rangle$$

$$\tau/\ \ \boldsymbol{X}|\boldsymbol{x}\rangle = \mathbf{x}|\boldsymbol{x}\rangle$$

$$\tau\boldsymbol{X}|\boldsymbol{x}\rangle = \boldsymbol{x}\tau|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x} + d\boldsymbol{x}\rangle$$

$$[\boldsymbol{X},\tau] = d\boldsymbol{x}|\boldsymbol{x} + d\boldsymbol{x}\rangle$$

- $|\boldsymbol{x}+d\boldsymbol{x}\rangle = |\boldsymbol{x}\rangle + \underbrace{d\boldsymbol{x}.\boldsymbol{\nabla}|\boldsymbol{x}\rangle}_{dx_j\nabla_j}$

- $$\begin{aligned}[\boldsymbol{X},\tau]|\boldsymbol{x}\rangle &\approx dx_i\big(|\boldsymbol{x}\rangle + dx_j\nabla_j|\boldsymbol{x}\rangle\big)\\ &\approx dx_i|\boldsymbol{x}\rangle + dx_i dx_j\nabla_j|\boldsymbol{x}\rangle \ldots\\ &\approx dx_i|\boldsymbol{x}\rangle + O(d\boldsymbol{x}^2)\end{aligned}$$

- $$\left.\begin{array}{l}[X_i,\underset{\overset{\downarrow}{I-id\boldsymbol{x}.\boldsymbol{K}}}{\tau}] \approx dx_i\end{array}\right\} \quad -idx_j[X_i,K_j]=dx_i = \delta_{ij}dx_j$$

$$\left.\begin{array}{l}\end{array}\right\} \quad [X_i,K_j] = i\delta_{ij}$$

$$\updownarrow$$

$$\boxed{K_i \equiv \frac{P_i}{\hbar}}$$

$$\tau(d\boldsymbol{x}) = I - \frac{i}{\hbar}\boldsymbol{P}.\,d\boldsymbol{x}$$

$$\Downarrow$$

$$[X_i, P_j] = i\hbar\delta_{ij}$$

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Week 7

Commutation Relations

Wave Functions in Position and Momentum Space

Recitation for MT

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Commutation Relations

$$[X_i, P_j] = i\hbar\delta_{ij}$$

$$[P_i, P_j] = 0$$

$$[X_i, X_j] = 0$$

- $[A,A] = 0$
- $[A,B] = -[B,A]$
- $[A,cI] = 0$
- $[A+B,C] = [A,C] + [B,C]$
- $[A,BC] = B[A,C] + [A,B]C$
- $[AB,C] = A[B,C] + [A,C]B$
- $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$

Wave Functions in Position and Momentum Space

- $|\psi\rangle$: Arbitrary State
- $X|x\rangle = x|x\rangle$
  - Complete ; $I = \int dx|x\rangle\langle x|$
  - Orthonormal ; $\langle x|x'\rangle = \delta(x - x')$

$$|\psi\rangle = (\int dx|x\rangle\langle x|)\,|\psi\rangle = \int dx|x\rangle\underbrace{\langle x|\,\psi\rangle}_{Coordinates\ \psi\text{(x)}}$$

$$\int dx|\langle x|\psi\rangle|^2 \rightarrow \langle\psi_1|\psi_2\rangle = \int dx\langle\psi_1|x\rangle\langle x|\,\psi_2\rangle = \int dx\langle x|\psi_1\rangle^*\langle x|\,\psi_2\rangle$$

$$\langle\psi_1|\psi_2\rangle = \int dx\psi_1(x)^*\psi_2(x)$$

Momentum Space Wave Function

- $P|p\rangle = p|p\rangle$
  - Complete ; $I = \int dp|p\rangle\langle p|$
  - Orthonormal ; $\langle p|p'\rangle = \delta(p - p')$

$$|\psi\rangle = \left(\int dp|p\rangle\langle p|\right)|\psi\rangle = \int dp|p\rangle\underbrace{\langle p|\,\psi\rangle}_{\psi(\mathrm{p})}$$

$$|\psi\rangle = \int dp|p\rangle\psi(\mathrm{p})$$

$\psi(\mathrm{x}) \leftrightarrow \psi(\mathrm{p})$

$\{|x\rangle\} \leftrightarrow \{|p\rangle\}$

$$|\psi\rangle = \int dx |x\rangle \psi(x)$$

$$\langle p| \ / \ |\psi\rangle = \int dx |x\rangle \psi(x)$$

$$\langle p|\psi\rangle = \int dx \langle p|x\rangle \psi(x)$$

$$\psi(p) = \int dx \langle p|x\rangle \psi(x)$$

$$|\psi\rangle = \int dp |p\rangle \psi(p)$$

$$\langle x| \ / \ |\psi\rangle = \int dp |p\rangle \psi(p)$$

$$\langle x|\psi\rangle = \int dp \langle x|p\rangle \psi(p)$$

$$\psi(x) = \int dp \langle x|p\rangle \psi(p)$$

$$\langle p|x\rangle = \langle x|p\rangle^*$$

To determine $\langle p|x\rangle$:

$$\langle p|\ /\ P|x\rangle = i\hbar\frac{d}{dx}|x\rangle$$

$$\langle p|P|x\rangle = i\hbar\langle p|\frac{d}{dx}|x\rangle$$

$$p\,\langle p|x\rangle = i\hbar\frac{d}{dx}\langle p|x\rangle$$

$$\frac{d}{dx}\langle p|x\rangle = -\frac{i}{\hbar}p\,\langle p|x\rangle$$

$$\langle p|x\rangle = \mathrm{N}\exp(-\frac{i}{\hbar}px)$$

$$\langle\psi|\psi\rangle = 1$$

$$|\psi\rangle = \int dx|x\rangle\psi(\mathrm{x})$$

$$\langle\psi|\psi\rangle \to \int dx|\psi(x)|^2 = 1$$

$$1 = \int dx\,\langle p|x\rangle\langle x|p\rangle$$

$$N = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\psi(\mathrm{p}) = \int dx \langle p|x\rangle \psi(\mathrm{x}) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \; \exp(-\frac{i}{\hbar}\boldsymbol{p.x})\, \psi(\mathrm{x})$$

$$\psi(\mathrm{x}) = \int dp \langle x|p\rangle \psi(\mathrm{p}) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \; \exp(\frac{i}{\hbar}\boldsymbol{p.x})\, \psi(\mathrm{p})$$

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Week 8

Quantum Dynamics

Time Evolution of Quantum Mechanical System

Dynamical Phases

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Time Evolution of Quantum Mechanical System

In the condition that the Hamiltonian is dependent on time, values of the same operator at different times correspond to different operators and are generally not commute.

Very special conditions are required to ensure that commuting property (adiabaticity).

$$i\left\{U\left(t,t_0\right)-U\left(t_0,t_0\right)\right\}=\int\limits_0^t dt^{'}H\left(t^{'}\right)U\left(t^{'},t_0\right)$$

$$U\left(t,t_0\right)=I-i\int\limits_0^t dt^{'}H\left(t^{'}\right)U\left(t^{'},t_0\right).$$

$$U^{(0)} = I.$$

$$\begin{aligned} U^{(1)} &= I - i \int\limits_{0}^{t} dt^{'} H\left(t^{'}\right) \\ U^{(2)} &= I + (-i) \int\limits_{0}^{t} dt^{'} H\left(t^{'}\right) + (-i)^{2} \int\limits_{0}^{t} dt^{'} \int\limits_{0}^{t^{'}} dt^{''} H\left(t^{'}\right) H\left(t^{''}\right) \end{aligned}$$

$$\boxed{U\left(t, t_{0}\right) = I + \sum (-i)^{n} \int\limits_{0}^{t} dt_{1} \int\limits_{0}^{t_{1}} dt_{2} ... \int\limits_{0}^{t_{n-1}} dt_{n} \left[H\left(t_{1}\right) ... H\left(t_{n}\right)\right]}$$

All time limits are converted to t.

The products in the integrant are converted to the time order product.

Put $1/n!$ in front of each term.

$$T\left[H(t_1)H(t_2)\right] = \begin{cases} H(t_1)H(t_2); & t_1 < t_2 \\ H(t_2)H(t_1); & t_2 < t_1 \end{cases}$$

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n T\left[H(t_1)\ldots H(t_n)\right] \tag{1}$$

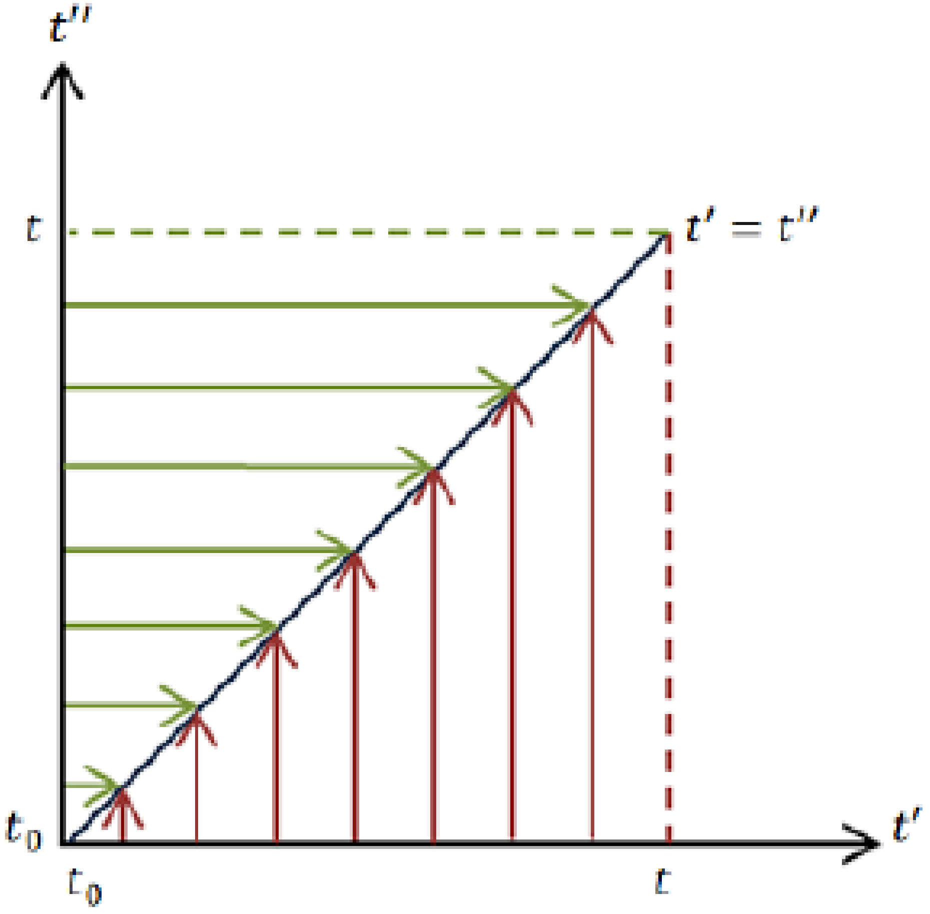
$$\boxed{U(t,t_0) = T\left[e^{-i\int_{t_0}^{t} dt' H(t')}\right]} \tag{2}$$

$$U\left(t,t_0\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{t_0}^{t}dt_1...\int\limits_{t_0}^{t}dt_nT\left[H\left(t_1\right)...H\left(t_n\right)\right] \tag{1}$$

$$U\left(t,t_0\right)=T\left[e^{-i\int_{t_0}^{t}dt^{'}H(t^{'})}\right] \tag{2}$$

$$\left[U\left(t,t_0\right)\right]_{(2,1)}=(-i)^2\int\limits_{t_0}^{t}dt^{'}\int\limits_{t_0}^{t^{'}}dt^{''}H\left(t^{'}\right)H\left(t^{''}\right) \tag{3}$$

$$\left[U\left(t,t_0\right)\right]_{(2,2)}=\frac{1}{2!}\int\limits_{t_0}^{t}dt^{'}\int\limits_{t_0}^{t^{'}}dt^{''}T\left[H\left(t^{'}\right)H\left(t^{''}\right)\right]. \tag{4}$$



- In the equation (3), first the integral of t'' must be taken
- The integration operation with the resulting t' must be done.
- In Figure, the first integral region is the region above the bisector.
- Considering that both fields are equal, the integral must be prefixed with a coefficient of ½
- Then, the time-ordered expression can be passed.
- Thus, equation (4) is obtained.
- Similar examples can be multiplied in other terms of the infinite sum.

Recitation:

- Starting from Equation (1), obtain the equation (2).

Dynamical Phases

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

$$\updownarrow$$

$$\left|\psi(t)\right\rangle = e^{-iHt}\left|\psi(0)\right\rangle$$

$$H = H\left(\mathbf{x}\left(t\right)\right)$$

$$H\left(t\right)\left|n(t)\right\rangle = E_n\left(t\right)\left|n(t)\right\rangle .$$

$$\left|\psi(t)\right\rangle = e^{i\phi_n}\left|n(t)\right\rangle$$

- Evolution Process
- External Force
- Interaction time
- No Physical Meaning

Dynamical Phases

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

$$\updownarrow$$

$$\left|\psi(t)\right\rangle = e^{-iHt}\left|\psi(0)\right\rangle$$

$$H\left(t\right)\left|n(t)\right\rangle = E_n\left(t\right)\left|n(t)\right\rangle .$$

$$\left|\psi(t)\right\rangle = e^{i\phi_n}\left|n(t)\right\rangle$$

$$\updownarrow$$

$$\left|n\left(\mathbf{x}(t)\right)\right\rangle = \left|n(t)\right\rangle$$

$$\theta_n(t) = -\int_0^t H_n(t^{'})dt^{'}$$

- Evolution Process
- External Force
- Interaction time
- No Physical Meaning

Dynamical Phases

$$i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$\updownarrow$$

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle .$$

$$|\psi(t)\rangle = e^{i\phi_n}|n(t)\rangle$$

$$\theta_n(t) = -\int_0^t H_n(t^{'})dt^{'}$$

- Evolution Process
- External Force
- Interaction time
- No Physical Meaning

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Week 9

Adiabatic Theorem

Quantum Mechanical Phases

Geometric Berry Phase

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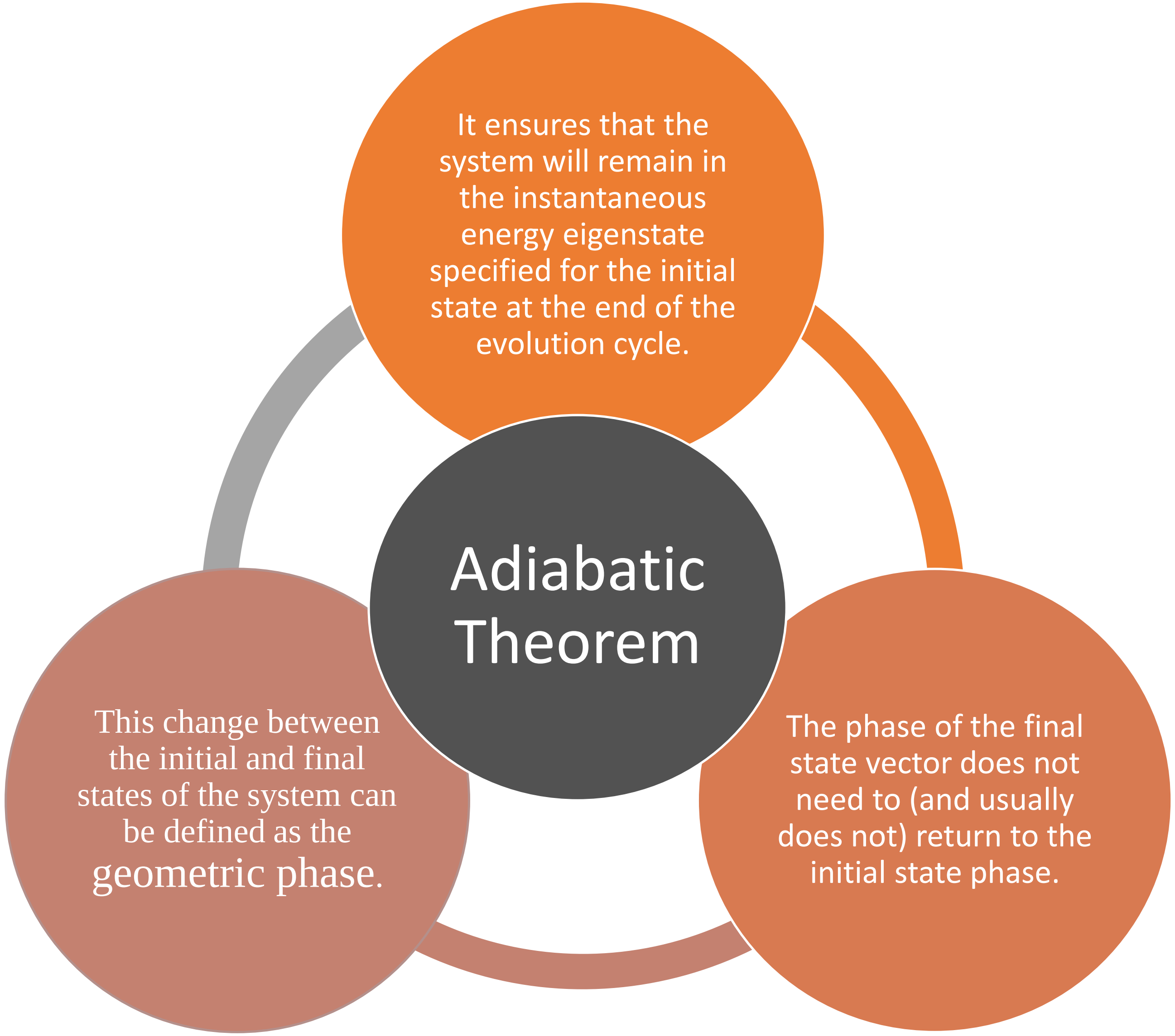
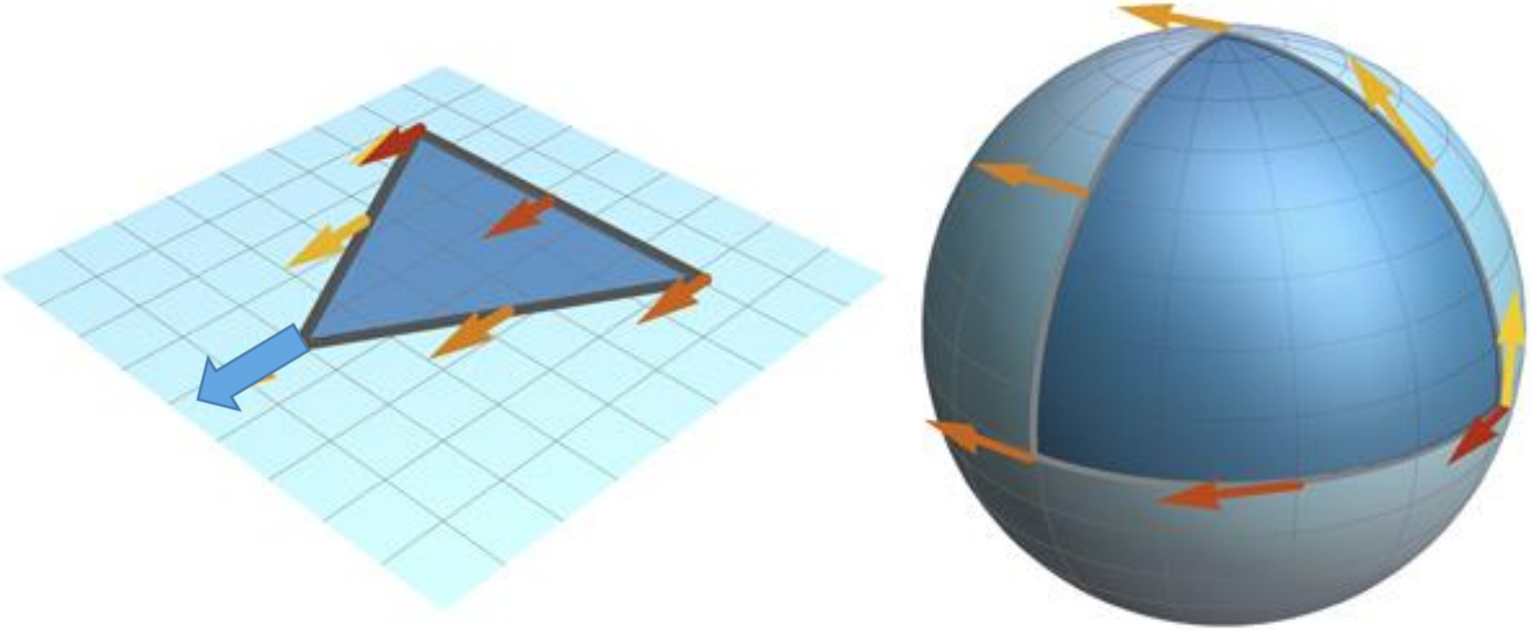


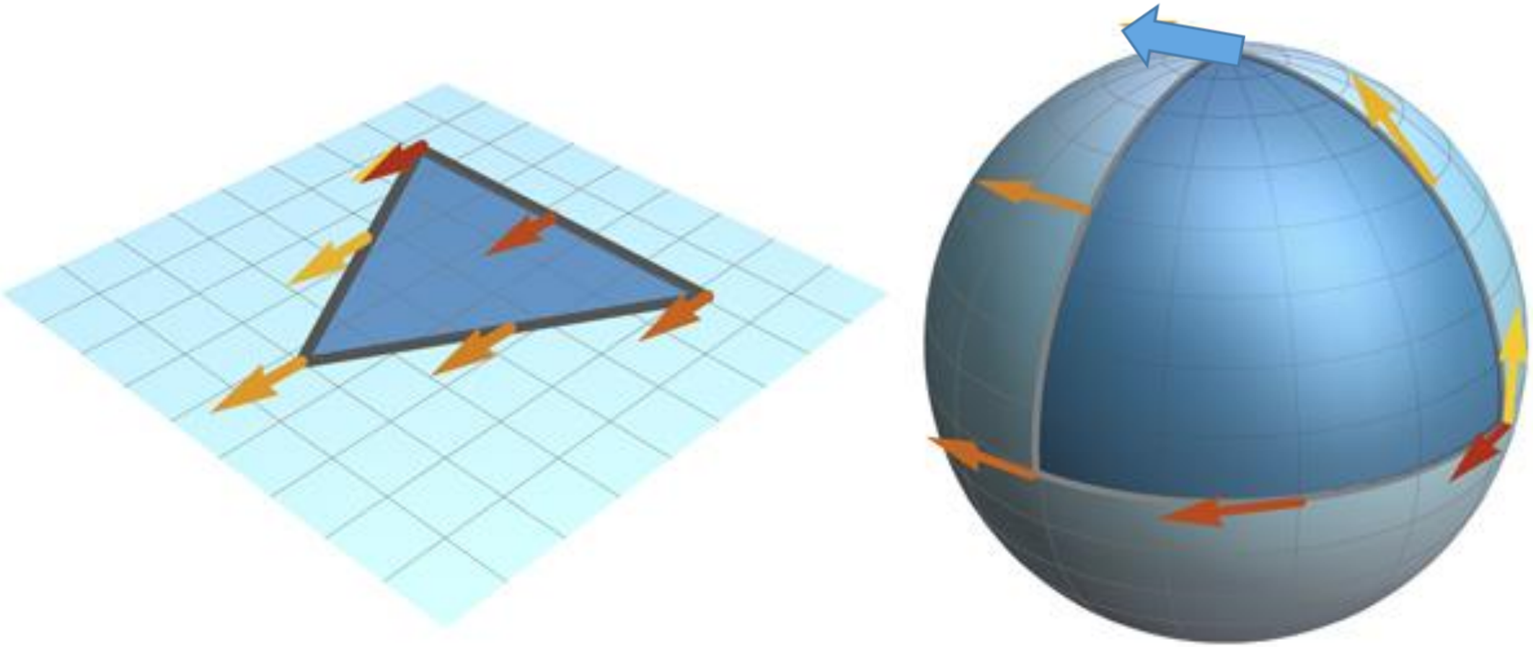
It ensures that the system will remain in the instantaneous energy eigenstate specified for the initial state at the end of the evolution cycle.

Adiabatic Theorem

The phase of the final state vector does not need to (and usually does not) return to the initial state phase.

This change between the initial and final states of the system can be defined as the geometric phase.





Addition to the known dynamical phase →

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

$$H\left(t\right)\left|n(t)\right\rangle = E_n\left(t\right)\left|n(t)\right\rangle .$$

$$\left|\psi(t)\right\rangle = e^{i\phi_n}\left|n(t)\right\rangle$$

$$\theta_n(t) = -\int_0^t H_n(t^{'})dt^{'}$$

- Due to the adiabatic nature of the process, wavefunction of the system gains an additional phase known as the **geometric phase** in the literature.

- Total phase that the system will gain, including geometric phase γ_n,

$$\phi_n = \theta_n + \gamma_n$$

$$i\left[\frac{d}{dt}e^{i(\theta_n+\gamma_n)}\right]|n(t)\rangle + ie^{i(\theta_n+\gamma_n)}\frac{d}{dt}|n(t)\rangle = H\left(t\right)e^{i\phi_n}|n(t)\rangle$$

$$i(i\dot{\theta}_n + i\dot{\gamma}_n)|n(t)\rangle + i\frac{d}{dt}|n(t)\rangle = H\left(t\right)|n(t)\rangle$$

$$i\langle n(t)|\frac{d}{dt}|n(t)\rangle - \langle n(t)|(\dot{\theta}_n + \dot{\gamma}_n)|n(t)\rangle = \langle n(t)|H\left(t\right)|n(t)\rangle .$$

$$\begin{aligned} LHS &= -\int_0^t \left\langle n(t')\right|\dot{\theta}_n\left|n(t')\right\rangle dt' - \int_0^t \left\langle n(t')\right|\dot{\gamma}_n\left|n(t')\right\rangle dt' \\ &\quad +i\int_0^t \left\langle n(t')\right|\frac{d}{dt'}\left|n(t')\right\rangle dt' \end{aligned}$$

$$RHS = \int_0^t \left\langle n(t')\right|H\left(t'\right)\left|n(t')\right\rangle dt'$$

$$\begin{aligned} LHS &= \int_0^t \left\langle n(t')\right|\left[-H\left(t'\right) + \frac{d}{dt'}\int_0^t H\left(t'\right)dt'\right]\left|n(t')\right\rangle dt' \\ RHS &= \int_0^t \left\langle n(t')\right|\dot{\gamma}_n\left|n(t')\right\rangle dt' - i\int_0^t \left\langle n(t')\right|\frac{d}{dt'}\left|n(t')\right\rangle dt' \end{aligned}$$

$$\gamma_n\left(t\right) - \gamma_n\left(0\right) = i\int_0^t \left\langle n(t')\right|\frac{d}{dt'}\left|n(t')\right\rangle dt'$$

$$\boxed{\gamma_n = i\oint_c \left\langle n(t)\middle|\boldsymbol{\nabla} n(t)\right\rangle . d\mathbf{x}}$$

$$\begin{aligned} 0 &= \boldsymbol{\nabla}\left\langle n(t) \middle| n(t) \right\rangle \\ &= \left\langle \boldsymbol{\nabla} n(t) \middle| n(t) \right\rangle + \left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle \\ &= \left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle^{*} + \left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle \\ &= 2\mathfrak{Re}\left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle . \end{aligned}$$

$$\boxed{\left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle = \mathbf{A}_n}$$

$$\oint_{c} \mathbf{F}.d\mathbf{l} = \iint_{s(c)} \boldsymbol{\nabla} \times \mathbf{F}.d\mathbf{S}$$

$$\gamma_n = i \oint\limits_{c} \mathbf{A_n}.d\mathbf{x} = i \iint\limits_{s(c)} \boldsymbol{\nabla} \times \mathbf{A}.d\mathbf{S}.$$

$$\begin{aligned} \varepsilon_{ijk}\nabla^j A^k &= \varepsilon_{ijk}\nabla^j \left\langle n(t) \middle| \nabla^k n(t) \right\rangle \\ &= \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| \nabla^k n(t) \right\rangle + \varepsilon_{ijk} \left\langle n(t) \middle| \nabla^j \nabla^k n(t) \right\rangle \end{aligned}$$

$$\gamma_n = i \iint\limits_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| \nabla^k n(t) \right\rangle dS_i$$

$$\sum_m \left| m \right\rangle \left\langle m \right| = I$$

$$\gamma_n = i \sum_m \iint\limits_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle dS_i$$

$$\begin{aligned}\gamma_n &= \iint\limits_{s(c)} \Big[i\varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| n \right\rangle \left\langle n \middle| \nabla^k n(t) \right\rangle \\ &\qquad + i \sum_{m \neq n} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle \Big] \, dS_i \\ &= i \sum_{m \neq n} \iint\limits_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle dS_i \\ &= -Im \sum_{m \neq n} \iint\limits_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle\end{aligned}$$

$$\left\langle m \middle| n \right\rangle = \delta_{mn} \quad \Longrightarrow \quad \begin{aligned} \boldsymbol{\nabla} \left\langle m \middle| n \right\rangle &= 0 \\ \left\langle \boldsymbol{\nabla} m \middle| n \right\rangle + \left\langle m \middle| \boldsymbol{\nabla} n \right\rangle &= 0 \\ \left\langle m \middle| \boldsymbol{\nabla} n \right\rangle &= -\left\langle n \middle| \boldsymbol{\nabla} m \right\rangle^* \end{aligned}$$

$$A_{mn}^{(i)} = \left\langle m \middle| \nabla^{i} n \right\rangle \quad \Rightarrow \quad \gamma_n = i \sum_{m \neq n} \iint_{s(c)} \varepsilon_{ijk} A_{mn}^{(j)*} A_{mn}^{(k)} dS_i$$

$$\begin{aligned}
\varepsilon_{ijk} A_{mn}^{(j)*} A_{mn}^{(k)} &= \varepsilon_{ijk} \left[R_{mn}^{(j)} - i \mathcal{J}_{mn}^{(j)} \right] \left[R_{mn}^{(k)} + i \mathcal{J}_{mn}^{(k)} \right] \\
&= \varepsilon_{ijk} \left[R_{mn}^{(j)} R_{mn}^{(k)} + \mathcal{J}_{mn}^{(j)} \mathcal{J}_{mn}^{(k)} \right] + i \varepsilon_{ijk} \left[R_{mn}^{(j)} \mathcal{J}_{mn}^{(k)} - \mathcal{J}_{mn}^{(j)} R_{mn}^{(k)} \right]
\end{aligned}$$

$$\begin{aligned}
\gamma_n &= -\sum_{m \neq n} \iint_{s(c)} \varepsilon_{ijk} \left[R_{mn}^{(j)} \mathcal{J}_{mn}^{(k)} - \mathcal{J}_{mn}^{(j)} R_{mn}^{(k)} \right] dS_i \\
&= -Im \sum_{m \neq n} \iint_{s(c)} \varepsilon_{ijk} \left\langle \nabla^{j} n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^{k} n(t) \right\rangle dS_i
\end{aligned}$$

$$[\nabla H\left(t\right)]\left|n(t)\right\rangle + H\left(t\right)\left|\nabla n(t)\right\rangle = [\nabla E_n\left(t\right)]\left|n(t)\right\rangle + E_n\left(t\right)\left|\nabla n(t)\right\rangle$$

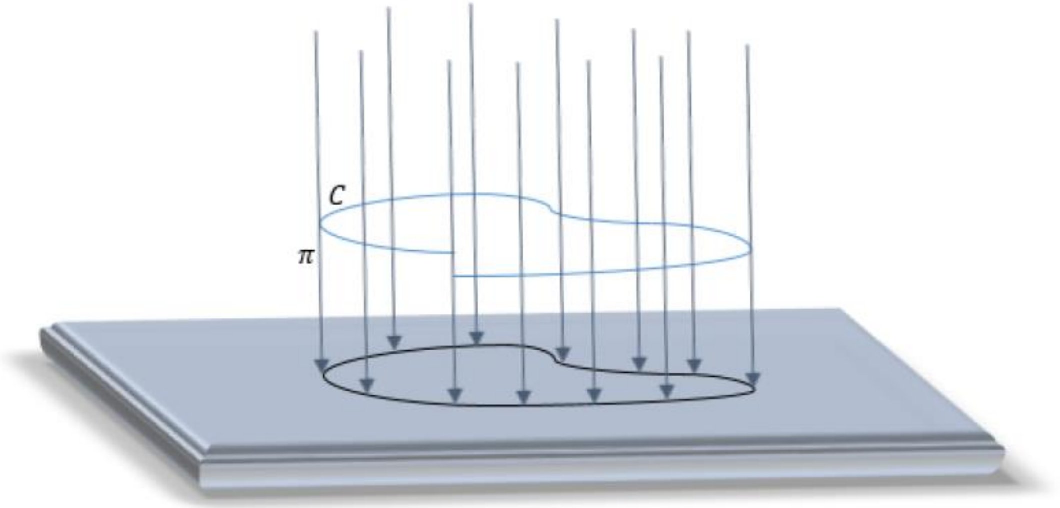
$$\left\langle m\right|\nabla H\left(t\right)\left|n(t)\right\rangle + E_m\left(t\right)\left\langle m\left|\nabla n(t)\right.\right\rangle = \nabla E_n\left(t\right)\left\langle m\left|n(t)\right.\right\rangle + E_n\left(t\right)\left\langle m\left|\nabla n(t)\right.\right\rangle$$

$$\left\langle m\left|\nabla n(t)\right.\right\rangle = \frac{\left\langle m\right|\boldsymbol{\nabla} H\left(t\right)\left|n(t)\right\rangle}{E_n - E_m}$$

$$\boxed{\gamma_n = -Im\sum_{m\neq n}\iint\limits_{s(c)}\varepsilon_{ijk}\frac{\left\langle n(t)\right|\nabla^j H\left(t\right)\left|m\right\rangle\left\langle m\right|\nabla^k H\left(t\right)\left|n(t)\right\rangle}{\left(E_n - E_m\right)^2}dS_i.}$$

Aharanov Anandan Phase

- Geometric Phase
- Projective Hilbert Space
- Adiabadic Approximation?
- Equivalance Classes
- Manifolds



Advanced Quantum Mechanics II

PEN425

Dr. H.Ozgur Cildiroglu

Advanced Quantum Mechanics II

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Week 10

Gauge Transformations

Ankara University | Physics Engineering Department

Dr. H. Ozgur Cildiroglu

Gauge Transformations

- Classical Mechanics
- Electromagnetic Theory
- Quantum Mechanics

Gauge Transformations

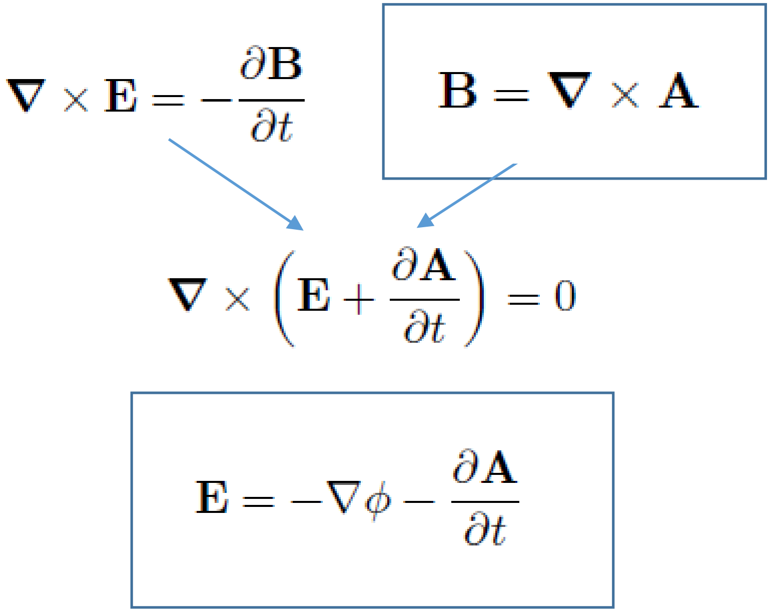
- Classical Mechanics
- Electromagnetic Theory
- Quantum Mechanics

$$\begin{tikzcd}
\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \arrow[dr] & & \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \arrow[dl] \\
& \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 &
\end{tikzcd}$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

Gauge Transformations

- Classical Mechanics
- Electromagnetic Theory
- Quantum Mechanics



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

$$\boxed{\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}}$$

Gauge Transformations

- Classical Mechanics
- Electromagnetic Theory
- Quantum Mechanics

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

$$\boxed{\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}}$$

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla f(\mathbf{x}, t)$$

$$\phi \to \phi' = \phi - \frac{\partial f(\mathbf{x}, t)}{\partial t}$$

Gauge Transformations in CM

$$\mathbf{F}_{Lorentz} = q\left[-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right)\right]$$

$$\mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right) = \nabla\left(\mathbf{v}.\mathbf{A}\right) - \left(\mathbf{v}.\nabla\right)\mathbf{A}$$

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \left(\mathbf{v}.\nabla\right)\mathbf{A}$$

$$\mathbf{F}_{Lorentz} = q\left[-\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\left(\mathbf{v}.\mathbf{A}\right) - \frac{d\mathbf{A}}{dt}\right]$$

$$\mathbf{F}_{Lorentz} = -\frac{\partial}{\partial \mathbf{x}}\left[q\phi - q\left(\mathbf{v}.\mathbf{A}\right)\right] + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\mathbf{x}}}\left(q\phi - q\left(\mathbf{v}.\mathbf{A}\right)\right)\right]$$

Gauge Transformations in CM

$$\mathbf{F}_{Lorentz} = q\left[-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \left(\boldsymbol{\nabla} \times \mathbf{A}\right)\right]$$

$$\mathbf{v} \times \left(\boldsymbol{\nabla} \times \mathbf{A}\right) = \nabla\left(\mathbf{v}.\mathbf{A}\right) - \left(\mathbf{v}.\nabla\right)\mathbf{A}$$

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \left(\mathbf{v}.\nabla\right)\mathbf{A}$$

$$\mathbf{F}_{Lorentz} = q\left[-\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\left(\mathbf{v}.\mathbf{A}\right) - \frac{d\mathbf{A}}{dt}\right]$$

$$\mathbf{F}_{Lorentz} = -\frac{\partial}{\partial \mathbf{x}}\left[q\phi - q\left(\mathbf{v}.\mathbf{A}\right)\right] + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\mathbf{x}}}\left(q\phi - q\left(\mathbf{v}.\mathbf{A}\right)\right)\right]$$

$$U = q\left[\phi - \left(\mathbf{v}.\mathbf{A}\right)\right]$$

$$\mathbf{F}_{Lorentz} = -\frac{\partial U}{\partial \mathbf{x}} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\mathbf{x}}}\right)$$

$$L = T - q\left[\phi + \left(\mathbf{v}.\mathbf{A}\right)\right]$$

$$H = \frac{1}{2}m\left(\mathbf{p} - q\mathbf{A}\right)^2 + q\phi.$$

$$\mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}$$

$$H \rightarrow H + q\phi$$

$$L^{'} = T - q\left[\phi^{'} - \left(\mathbf{v}.\mathbf{A}^{'}\right)\right]$$

$$L^{'} = T - q\left[\phi - (\mathbf{v}.\mathbf{A})\right] + q\frac{\partial f}{\partial t} + q\mathbf{v}.\boldsymbol{\nabla} f$$

$$\frac{df}{dt} = \mathbf{v}.\nabla f + \frac{\partial f}{\partial t}$$

$$\boxed{\begin{gathered} L^{'} = L + q\frac{df}{dt} \\ H^{'} = H - \frac{\partial F}{\partial t} \end{gathered}}$$

In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$\begin{aligned} p^0 &\rightarrow p^0 - eA^0 \\ \mathbf{p} &\rightarrow \mathbf{p} - e\mathbf{A} \end{aligned}$$

In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$\begin{aligned} p^0 &\rightarrow p^0 - eA^0 \\ \mathbf{p} &\rightarrow \mathbf{p} - e\mathbf{A} \end{aligned}$$

$$\begin{aligned} i\partial_t \rightarrow i\partial_t - e\phi &= i\left[\partial_t + ie\phi\right] = iD_t \\ -i\nabla \rightarrow -i\boldsymbol{\nabla} - e\mathbf{A} &= -i\left[\boldsymbol{\nabla} - ie\mathbf{A}\right] = -i\mathbf{D}. \end{aligned}$$

$$\begin{aligned} D_t &= \partial_t + ie\phi \\ \mathbf{D} &= \boldsymbol{\nabla} - ie\mathbf{A} \end{aligned}$$

In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$\begin{aligned} p^0 &\rightarrow p^0 - eA^0 \\ \mathbf{p} &\rightarrow \mathbf{p} - e\mathbf{A} \end{aligned}$$

$$\begin{aligned} i\partial_t \rightarrow i\partial_t - e\phi &= i\left[\partial_t + ie\phi\right] = iD_t \\ -i\nabla \rightarrow -i\boldsymbol{\nabla} - e\mathbf{A} &= -i\left[\boldsymbol{\nabla} - ie\mathbf{A}\right] = -i\mathbf{D}. \end{aligned}$$

$$\begin{aligned} D_t &= \partial_t + ie\phi \\ \mathbf{D} &= \boldsymbol{\nabla} - ie\mathbf{A} \end{aligned}$$

$$\left[iD_t + \frac{1}{2m}D^2\right]\psi = 0.$$

$$\left[iD_t' + \frac{1}{2m}D'^2\right]\psi' = 0.$$

In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$\begin{aligned} p^0 &\rightarrow p^0 - eA^0 \\ \mathbf{p} &\rightarrow \mathbf{p} - e\mathbf{A} \end{aligned}$$

$$\begin{aligned} i\partial_t \rightarrow i\partial_t - e\phi &= i\left[\partial_t + ie\phi\right] = iD_t \\ -i\nabla \rightarrow -i\boldsymbol{\nabla} - e\mathbf{A} &= -i\left[\boldsymbol{\nabla} - ie\mathbf{A}\right] = -i\mathbf{D}. \end{aligned}$$

$$\begin{aligned} D_t &= \partial_t + ie\phi \\ \mathbf{D} &= \boldsymbol{\nabla} - ie\mathbf{A} \end{aligned}$$

$$\left[iD_t + \frac{1}{2m}D^2\right]\psi = 0.$$

$$\left[iD_t' + \frac{1}{2m}D'^2\right]\psi' = 0.$$

$$\psi \rightarrow \psi' = U\psi$$

In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$\begin{aligned} p^0 &\rightarrow p^0 - eA^0 \\ \mathbf{p} &\rightarrow \mathbf{p} - e\mathbf{A} \end{aligned}$$

$$\begin{aligned} i\partial_t \rightarrow i\partial_t - e\phi &= i\left[\partial_t + ie\phi\right] = iD_t \\ -i\nabla \rightarrow -i\boldsymbol{\nabla} - e\mathbf{A} &= -i\left[\boldsymbol{\nabla} - ie\mathbf{A}\right] = -i\mathbf{D}. \end{aligned}$$

$$\begin{aligned} D_t &= \partial_t + ie\phi \\ \mathbf{D} &= \boldsymbol{\nabla} - ie\mathbf{A} \end{aligned}$$

$$\left[iD_t + \frac{1}{2m}D^2\right]\psi = 0.$$

$$\left[iD_t^{'} + \frac{1}{2m}D'^2\right]\psi^{'} = 0.$$

$$\psi \rightarrow \psi^{'} = U\psi$$

$$\begin{aligned} D_t\psi \rightarrow \left(D_t\psi\right)^{'} &= D_t^{'}\psi^{'} &= U\left(D_t\psi\right) \\ \mathbf{D}\psi \rightarrow \left(\mathbf{D}\psi\right)^{'} &= \mathbf{D}^{'}\psi^{'} &= U\left(\mathbf{D}\psi\right) \end{aligned}$$

In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$\begin{aligned} p^0 &\rightarrow p^0 - eA^0 \\ \mathbf{p} &\rightarrow \mathbf{p} - e\mathbf{A} \end{aligned}$$

$$\begin{aligned} i\partial_t \rightarrow i\partial_t - e\phi &= i\left[\partial_t + ie\phi\right] = iD_t \\ -i\nabla \rightarrow -i\boldsymbol{\nabla} - e\mathbf{A} &= -i\left[\boldsymbol{\nabla} - ie\mathbf{A}\right] = -i\mathbf{D}. \end{aligned}$$

$$\begin{aligned} D_t &= \partial_t + ie\phi \\ \mathbf{D} &= \boldsymbol{\nabla} - ie\mathbf{A} \end{aligned}$$

$$\left[iD_t + \frac{1}{2m}D^2\right]\psi = 0.$$

$$\left[iD_t^{'} + \frac{1}{2m}D^{\prime 2}\right]\psi^{'} = 0.$$

$$\psi \rightarrow \psi^{'} = U\psi$$

$$\begin{aligned} D_t\psi \rightarrow (D_t\psi)^{'} &= D_t^{'}\psi^{'} &= U\left(D_t\psi\right) \\ \mathbf{D}\psi \rightarrow (\mathbf{D}\psi)^{'} &= \mathbf{D}^{'}\psi^{'} &= U\left(\mathbf{D}\psi\right) \end{aligned}$$

$$U = e^{ieg(\mathbf{x},t)}$$

$$\begin{aligned} D_t^{'} &= \partial_t + ie\phi^{'} \\ \mathbf{D}^{'} &= \boldsymbol{\nabla} - ie\mathbf{A}^{'} \end{aligned}$$

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$\begin{aligned} p^0 &\to p^0 - eA^0 \\ \mathbf{p} &\to \mathbf{p} - e\mathbf{A} \end{aligned}$$

$$\begin{aligned} i\partial_t \to i\partial_t - e\phi &= i\left[\partial_t + ie\phi\right] = iD_t \\ -i\nabla \to -i\boldsymbol{\nabla} - e\mathbf{A} &= -i\left[\boldsymbol{\nabla} - ie\mathbf{A}\right] = -i\mathbf{D}. \end{aligned}$$

$$\begin{aligned} D_t &= \partial_t + ie\phi \\ \mathbf{D} &= \boldsymbol{\nabla} - ie\mathbf{A} \end{aligned}$$

$$\left[iD_t + \frac{1}{2m}D^2\right]\psi = 0.$$

$$\left[iD_t^{'} + \frac{1}{2m}D^{'2}\right]\psi^{'} = 0.$$

$$\begin{aligned} D_t\psi \to \left(D_t\psi\right)^{'} &= D_t^{'}\psi^{'} &= U\left(D_t\psi\right) \\ \mathbf{D}\psi \to \left(\mathbf{D}\psi\right)^{'} &= \mathbf{D}^{'}\psi^{'} &= U\left(\mathbf{D}\psi\right) \end{aligned}$$

$$U = e^{ieg(\mathbf{x},t)}$$

$$\begin{aligned} D_t^{'} &= \partial_t + ie\phi^{'} \\ \mathbf{D}^{'} &= \boldsymbol{\nabla} - ie\mathbf{A}^{'} \end{aligned}$$

$$\left[\boldsymbol{\nabla} - ie\mathbf{A}^{'}\right]U\psi = U\left[\boldsymbol{\nabla} - ie\mathbf{A}\right]\psi$$

$$U\boldsymbol{\nabla}\psi + \left(\boldsymbol{\nabla}U\right)\psi - ie\mathbf{A}^{'}U\psi = U\boldsymbol{\nabla}\psi - ie\mathbf{A}U\psi$$

$$\left[\left(\mathbf{A}^{'} - \mathbf{A}\right)U + \frac{i}{e}\boldsymbol{\nabla}U\right]\psi = 0$$

$$\mathbf{A}^{'} = \mathbf{A} - \frac{i}{e}\left(\boldsymbol{\nabla}U\right)U^{-1}$$

$$\left(\boldsymbol{\nabla}U\right)U^{-1} = ie\boldsymbol{\nabla}g(\mathbf{x},t)$$

$$\boxed{\mathbf{A}^{'} = \mathbf{A} + \boldsymbol{\nabla}g(\mathbf{x},t).}$$

Recitation

- Show that $\phi' = \phi - \frac{\partial g}{\partial t}$.

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Advanced Quantum Mechanics II

PEN425

Dr. H.Ozgur Cildiroglu

Advanced Quantum Mechanics II

PEN425

Week 11

Topological Phases

Aharonov-Bohm Phase

Aharonov-Casher Phase

He-Mc-Kellar-Wilkens Phase

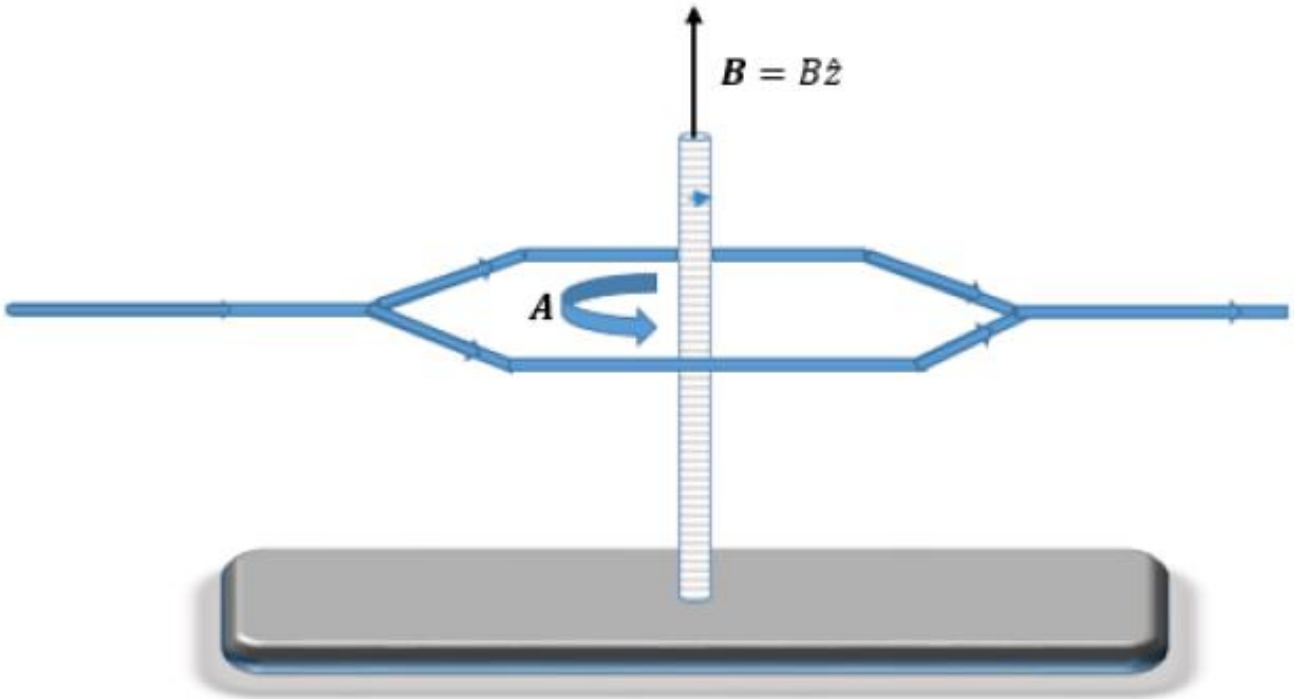
Ankara University | Physics Engineering Department

Dr. H. Ozgur Cildiroglu

Topological Phases

- Classical Force X
- Multiply-Connected Region
- Vector Potential
- Controlled by Physical Quantity in Singular Region

Aharonov - Bohm Phase



$$\psi' = e^{-ie\oint \mathbf{A}.d\mathbf{l}}\psi$$

$$\mathbf{A}' = \mathbf{A} + \nabla g(\mathbf{x},t).$$

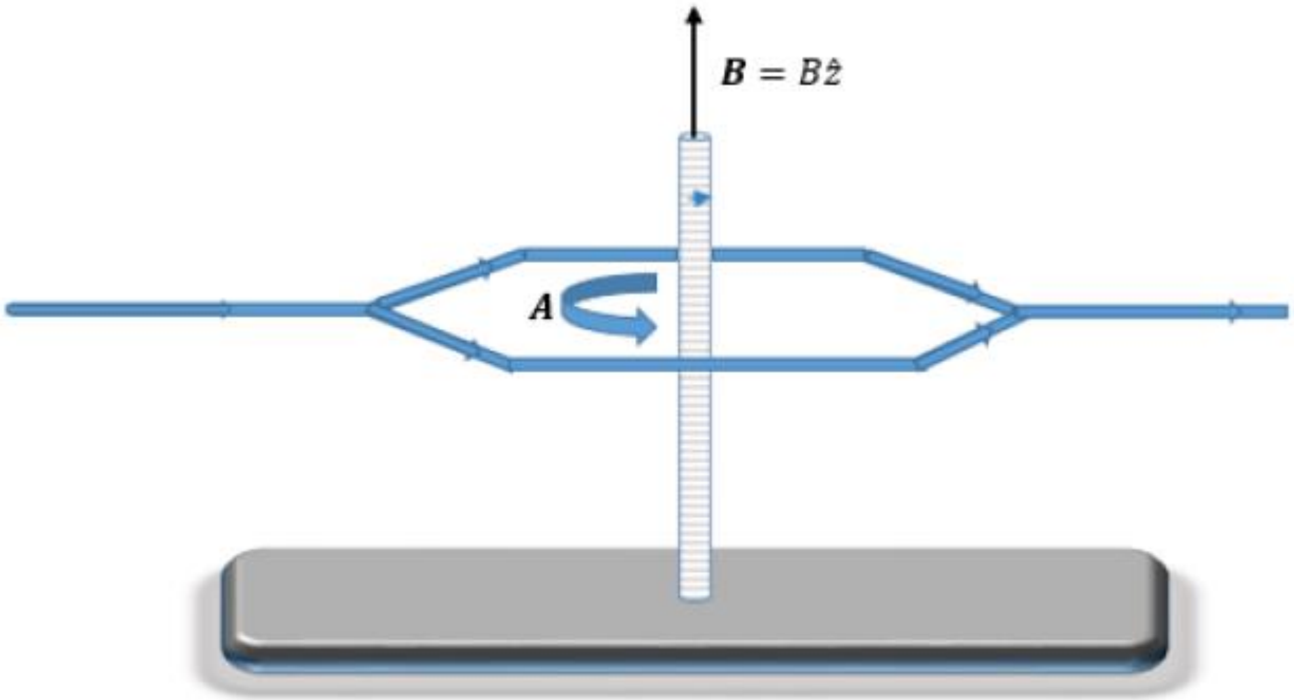
$$\mathbf{A}' = 0$$

$$\mathbf{A} = -\nabla g(\mathbf{x},t)$$

$$g(\mathbf{x},t) = -\int^{\mathbf{x}} \mathbf{A}\left(\mathbf{x}'\right).d\mathbf{x}'$$

$$\psi \to \psi' = U\psi \qquad U = e^{ieg(\mathbf{x},t)}$$

Aharonov - Bohm Phase



$$\mathbf{F}_{Lorentz} = q\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right]$$

$$\mathbf{A}^{'} = \mathbf{A} + \boldsymbol{\nabla} g(\mathbf{x}, t).$$

$$\mathbf{A}' = 0$$

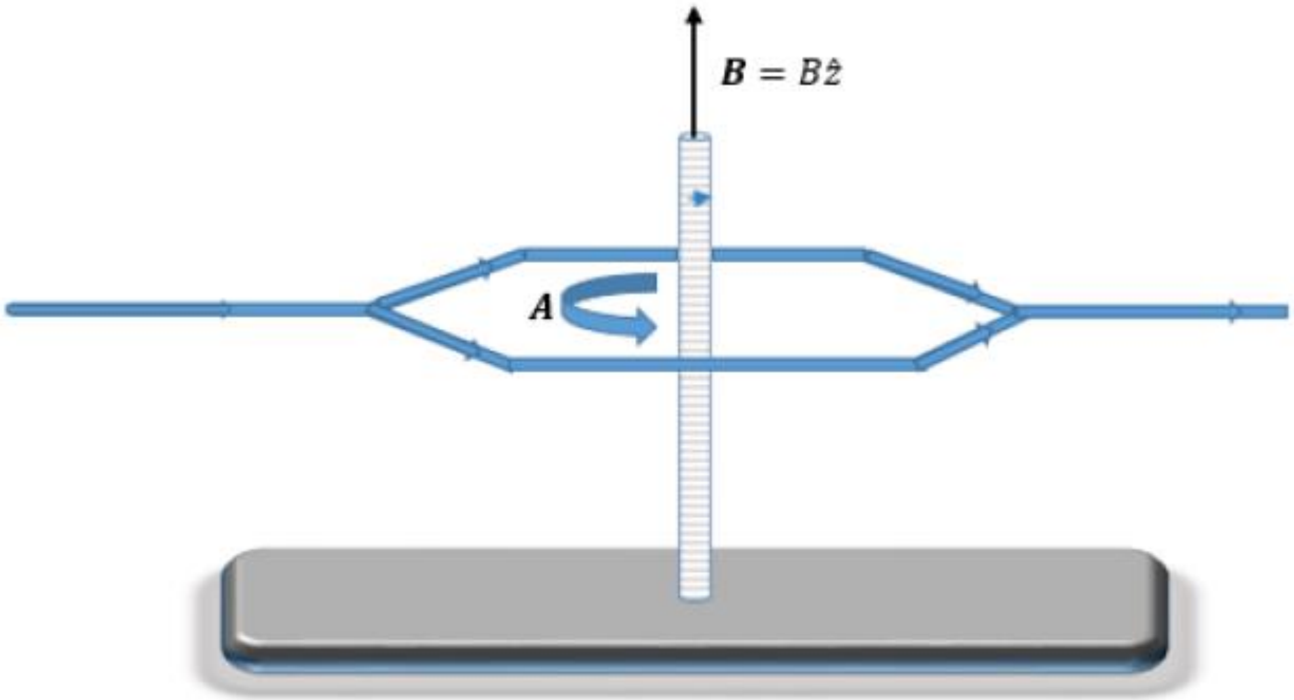
$$\mathbf{A} = -\boldsymbol{\nabla} g(\mathbf{x}, t)$$

$$g(\mathbf{x}, t) = -\int^{\mathbf{x}} \mathbf{A}\left(\mathbf{x}^{'}\right) . d\mathbf{x}^{'}$$

$$\psi \rightarrow \psi^{'} = U\psi \qquad U = e^{ieg(\mathbf{x},t)}$$

$$\psi' = e^{-ie\int^{x} \mathbf{A}.\mathbf{dl}} \psi$$

Aharonov - Bohm Phase



$$\psi' = e^{-ie\int^{x}\mathbf{A}.d\mathbf{l}}\psi$$

$$\begin{aligned}\psi &= \psi_1 + \psi_2 \\ &= U_{(1)}\psi_1^{'} + U_{(2)}\psi_2^{'}.\end{aligned}$$

$$\psi = e^{-ie\int_{l_1}\mathbf{A}.d\mathbf{l}}\psi_1^{'} + e^{-ie\int_{l_2}\mathbf{A}.d\mathbf{l}}\psi_2^{'}$$

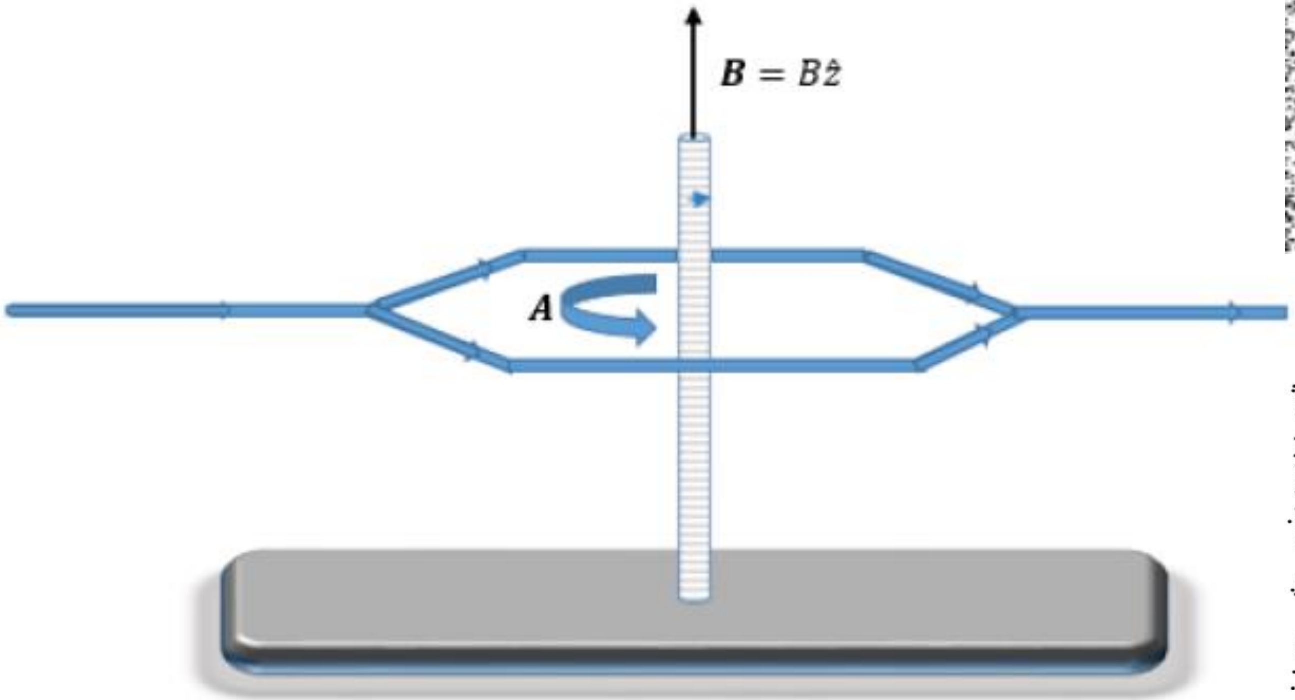
$$\psi = e^{-ie\int_{l_1}\mathbf{A}.d\mathbf{l}}\left[\psi_1^{'} + e^{ie\left(\int_{l_1}\mathbf{A}.d\mathbf{l} - \int_{l_2}\mathbf{A}.d\mathbf{l}\right)}\psi_2^{'}\right]$$

$$\int_{l_1}\mathbf{A}.d\mathbf{l} - \int_{l_2}\mathbf{A}.d\mathbf{l} = \int_{l_1}\mathbf{A}.d\mathbf{l} + \int_{l_2^{'}}\mathbf{A}.d\mathbf{l} = \oint \mathbf{A}.d\mathbf{l}$$

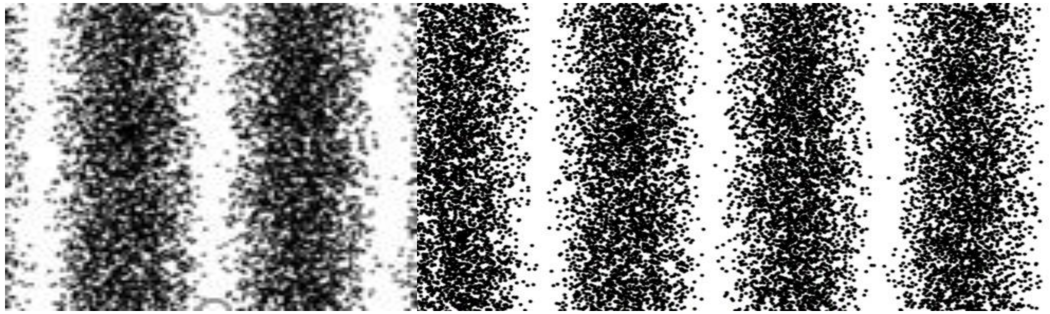
$$\psi = e^{-ie\int_{l_1}\mathbf{A}.d\mathbf{l}}\left[\psi_1^{'} + e^{ie\oint\mathbf{A}.d\mathbf{l}}\psi_2^{'}\right]$$

$$\delta = e\oint \mathbf{A}.d\mathbf{l}$$

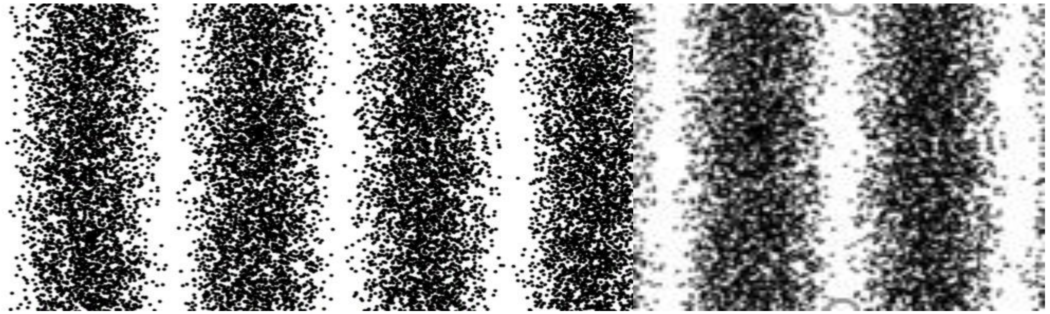
Aharonov - Bohm Phase



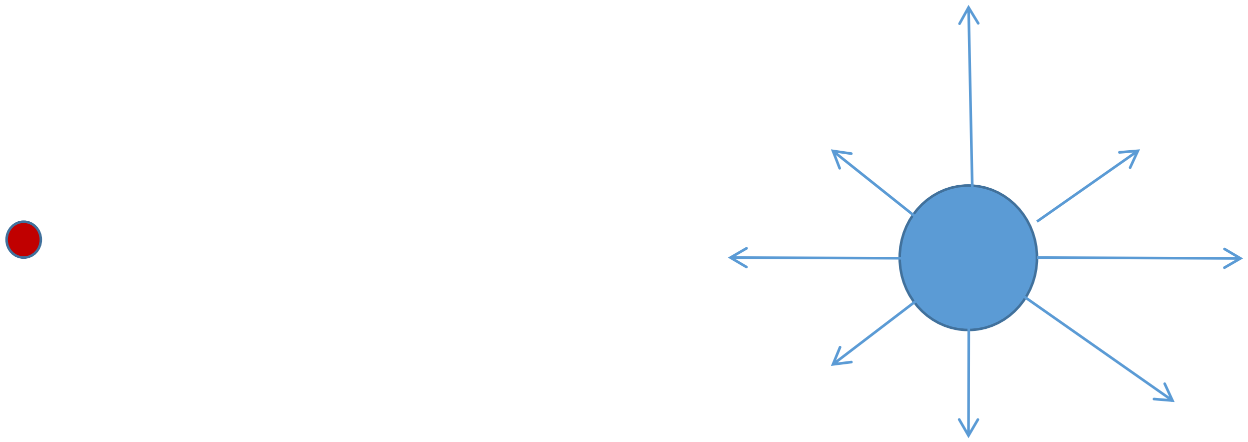
$\boldsymbol{B} = 0$



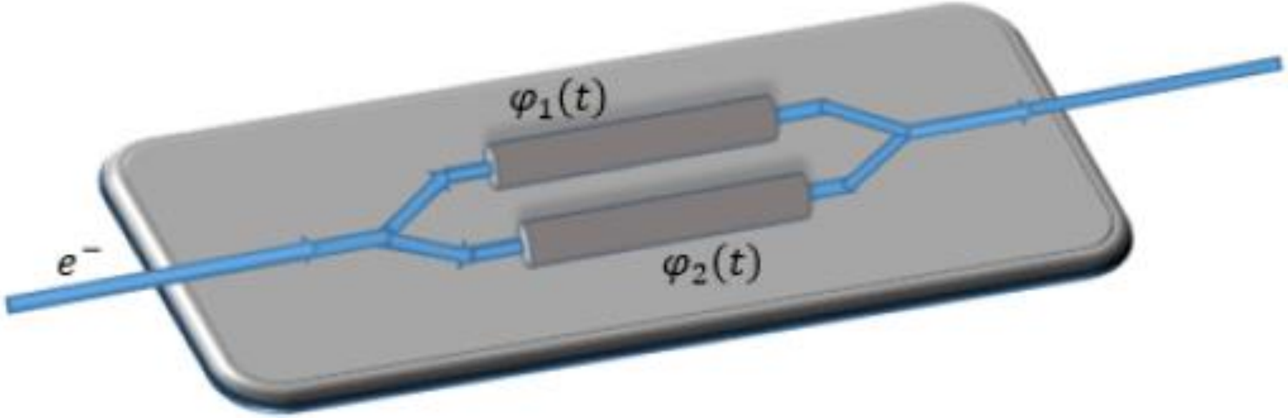
$\boldsymbol{B} \neq 0$



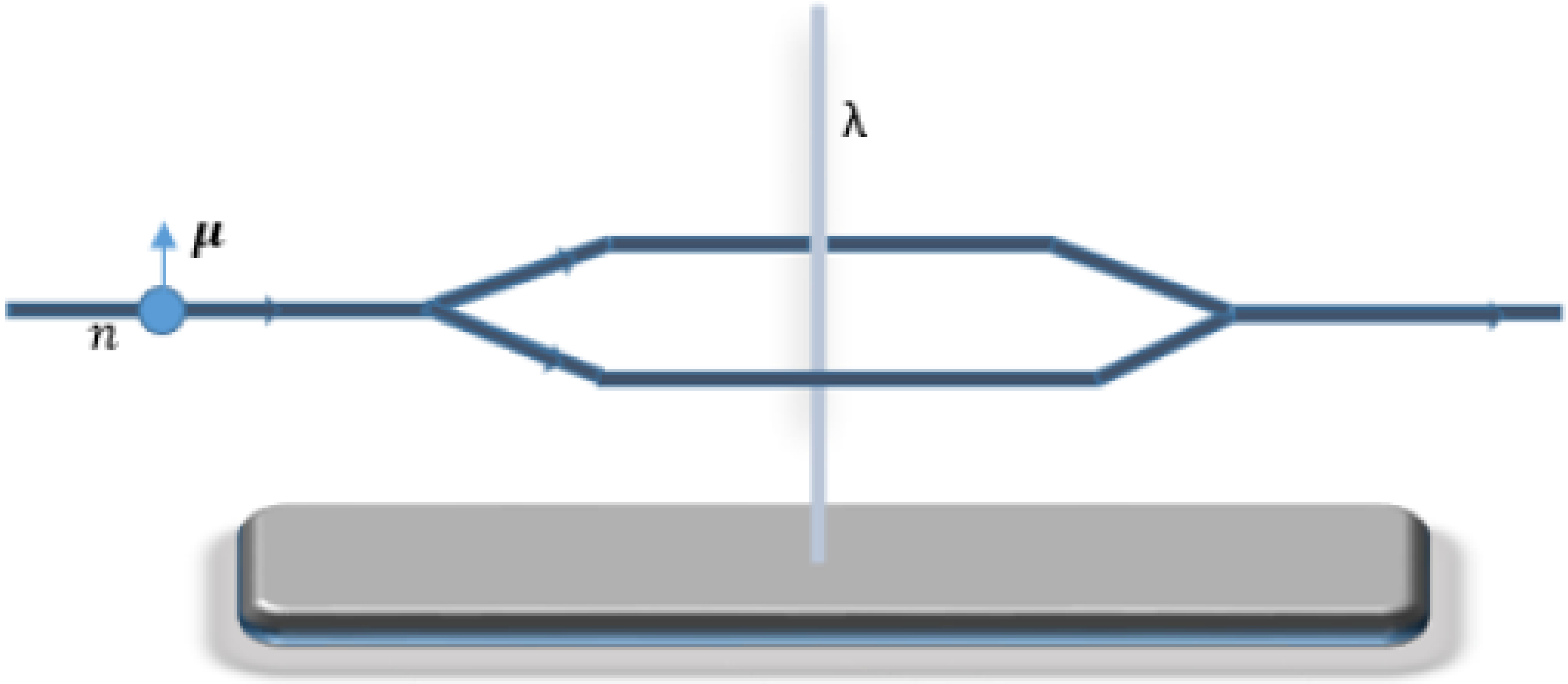
$$\psi' = \; e^{-ie\Phi}\psi.$$



Scalar AB Effect

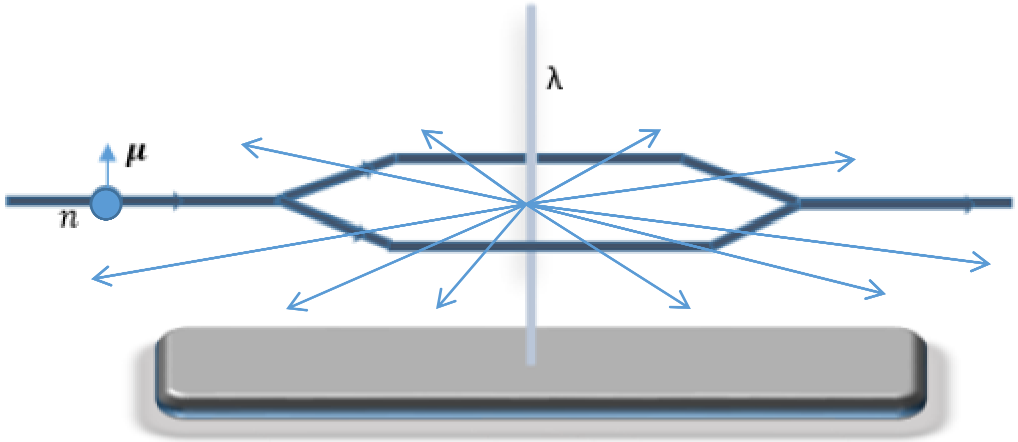


Recall: Dynamical Phases



Aharonov-Casher Effect

Classical Force?



$$m\dot{v}_i = e\left[\partial_j A_i\right] V_j \neq 0.$$

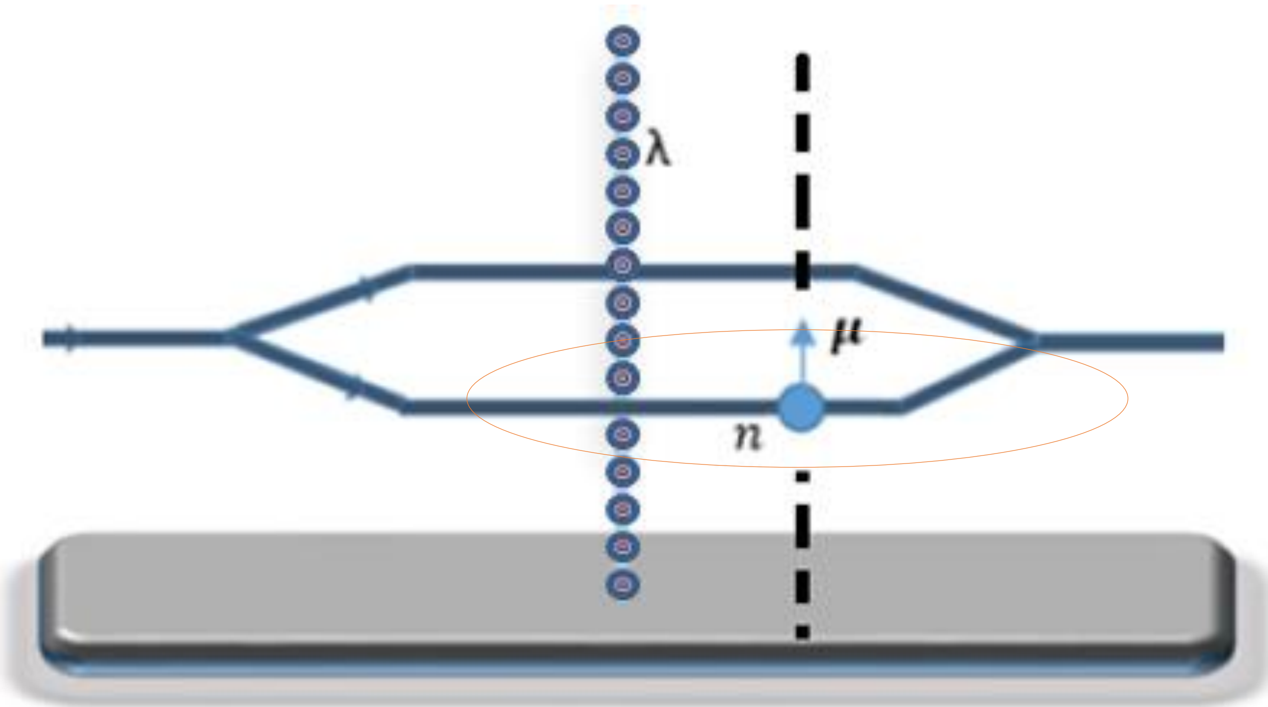
$$L = \frac{m\mathbf{v}^2}{2} + \frac{M\mathbf{V}^2}{2} - e\mathbf{A}.(\mathbf{v} - \mathbf{V})$$

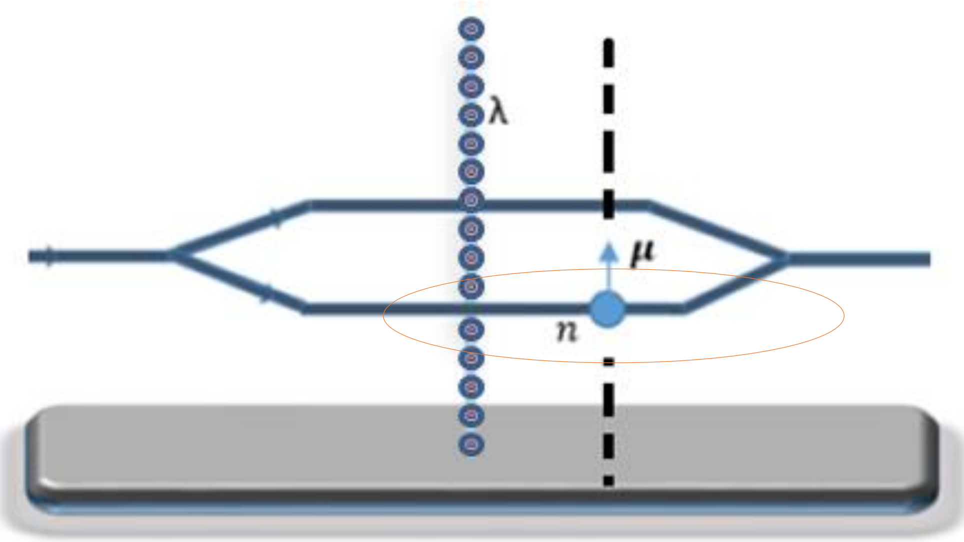
$$\begin{aligned} m\dot{v}_i &= e\frac{\partial A_i}{\partial r_j}(v_j - V_j) - e\frac{\partial A_j}{\partial r_i}(v_j - V_j) \\ &= e\left[\frac{\partial A_i}{\partial r_j} - \frac{\partial A_j}{\partial r_i}\right](v_j - V_j) \\ &= -e\left[\partial_i A_j - \partial_j A_i\right](v - V)_j \\ &= -e\varepsilon_{ijk}(v - V)_j B_k = 0. \end{aligned}$$

$$L = \frac{mv^2}{2} + \frac{MV^2}{2} + e\mathbf{A}.\mathbf{v}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = \frac{\partial L}{\partial r_i}$$

$$m\dot{v}_i = e\varepsilon_{ijk}v_j B_k + e\left[\partial_j A_i\right] V_j$$


λ
μ
n



$$L = \frac{MV^2}{2} + e\mathbf{A}.\mathbf{V}$$

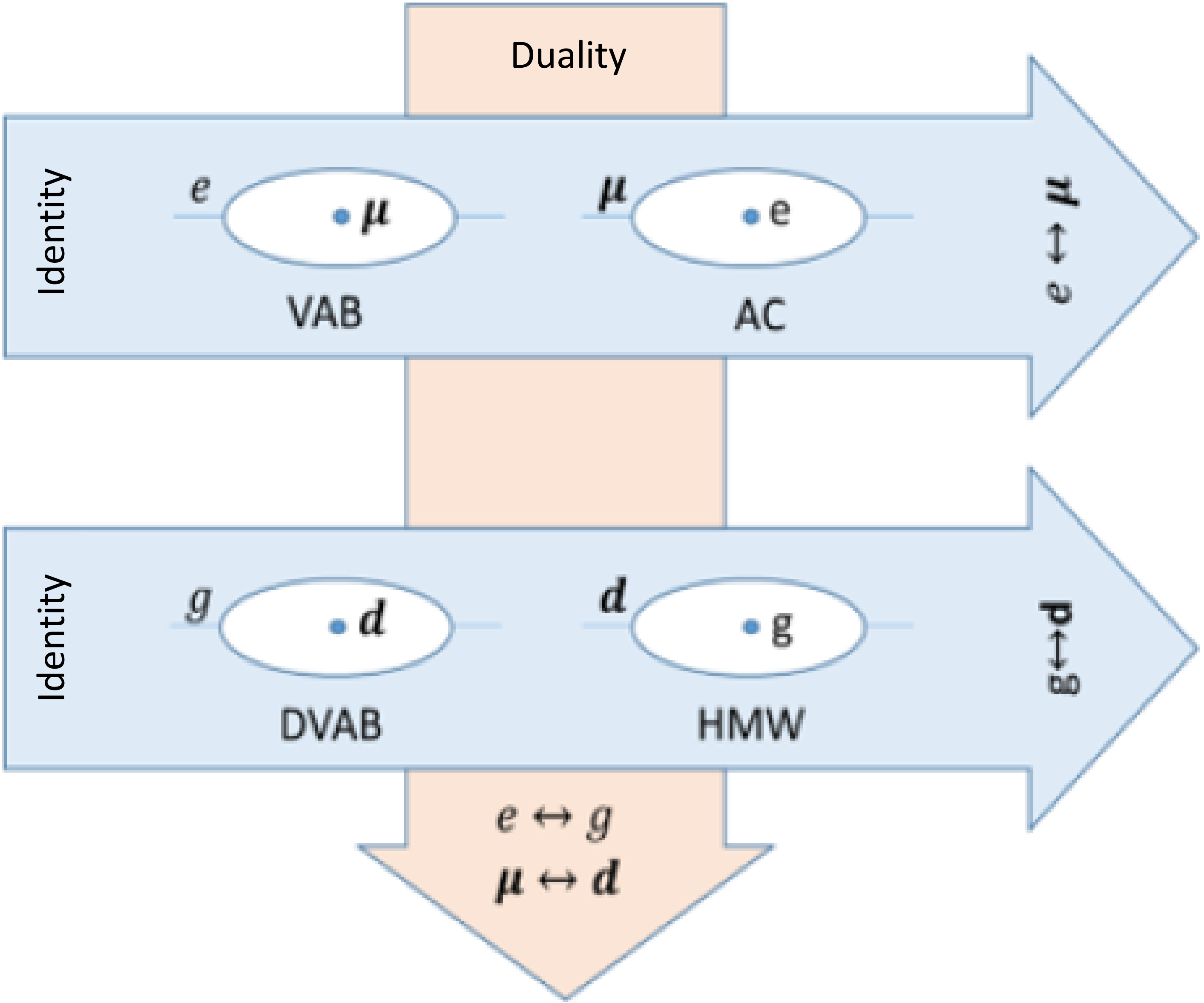
$$H = \dot{\mathbf{x}}.\mathbf{p} - L$$

$$H = \frac{[\mathbf{p} - e\mathbf{A}]^2}{2m}$$

$$\mathbf{p} = \frac{\partial L}{\partial \dot{x}} = M\mathbf{V} + e\mathbf{A}$$

$$\delta = \oint \mathbf{A}.d\mathbf{R}$$

$$\psi' = e^{-ie\oint \mathbf{A}.d\mathbf{R}}\psi$$



Duality

Identity

e $\bullet\,\boldsymbol{\mu}$

VAB

$\boldsymbol{\mu}$ $\bullet\,$e

AC

$e \leftrightarrow \boldsymbol{\mu}$

Identity

g $\bullet\,\boldsymbol{d}$

DVAB

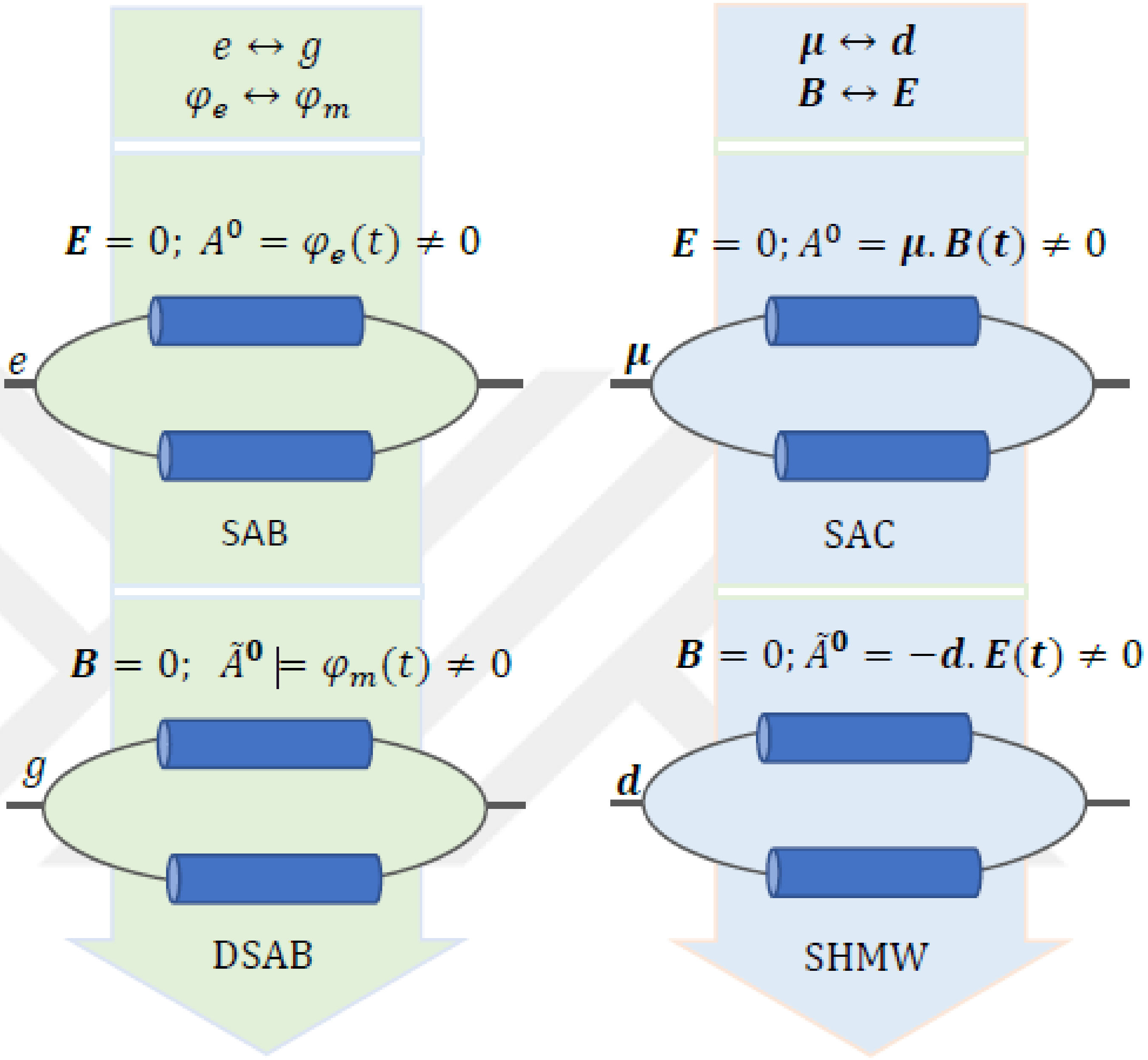
$\boldsymbol{d}$ $\bullet\,$g

HMW

g$\leftrightarrow\boldsymbol{d}$

$e \leftrightarrow g$

$\boldsymbol{\mu} \leftrightarrow \boldsymbol{d}$


$e \leftrightarrow g$
$\varphi_e \leftrightarrow \varphi_m$
$\boldsymbol{E} = 0;\ A^{\mathbf{0}} = \varphi_e(t) \neq 0$
e
SAB
$\boldsymbol{B} = 0;\ \tilde{A}^{\mathbf{0}} |= \varphi_m(t) \neq 0$
g
DSAB
$\boldsymbol{\mu} \leftrightarrow \boldsymbol{d}$
$\boldsymbol{B} \leftrightarrow \boldsymbol{E}$
$\boldsymbol{E} = 0; A^{\mathbf{0}} = \boldsymbol{\mu}.\boldsymbol{B}(\boldsymbol{t}) \neq 0$
$\boldsymbol{\mu}$
SAC
$\boldsymbol{B} = 0; \tilde{A}^{\mathbf{0}} = -\boldsymbol{d}.\boldsymbol{E}(\boldsymbol{t}) \neq 0$
$\boldsymbol{d}$
SHMW

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Week 12

Schrödinger Picture

Heisenberg Picture

Dirac Picture

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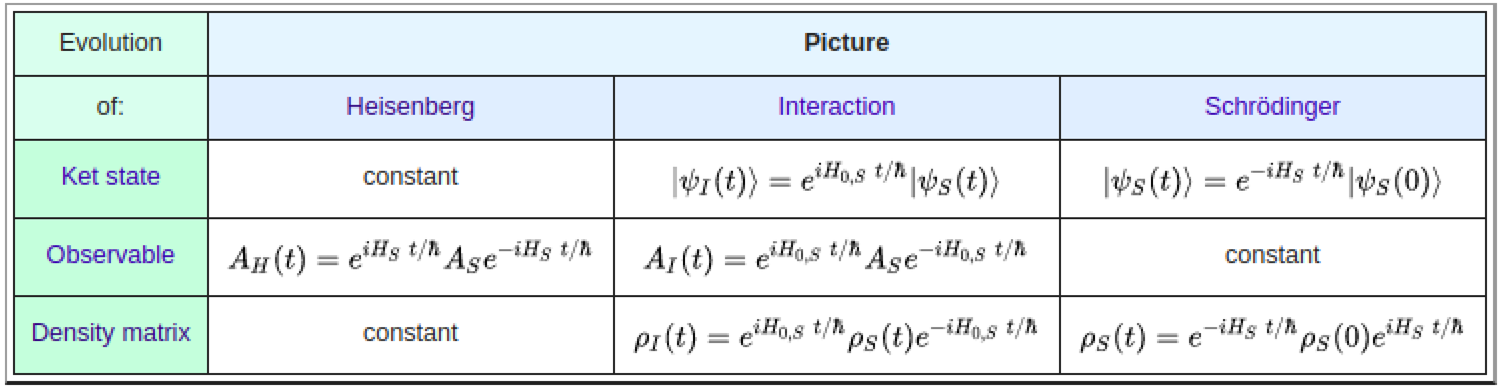
Dr. H. Ozgur Cildiroglu

Early Questions

- Interpretation of Quantum Mechanics
  - Mean?
  - Calculations?
  - State?
  - Measurement?
  - Evolution?
  -
  -

3 Different Answers

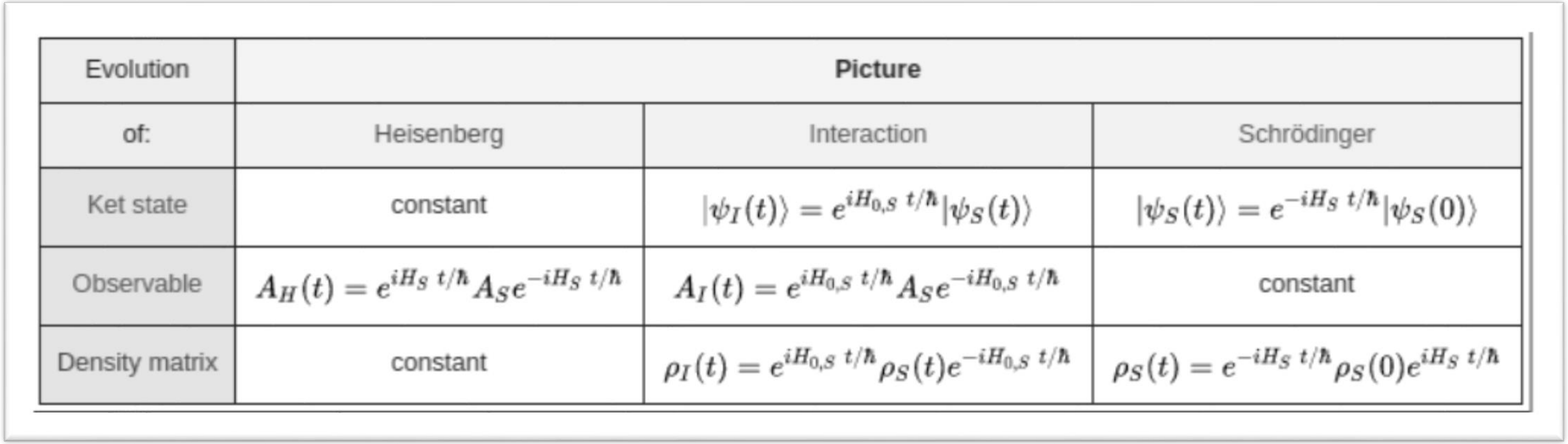
Pictures of Quantum Mechanics



Evolution	Picture		
of:	Heisenberg	Interaction	Schrödinger
Ket state	constant	$\vert\psi_I(t)\rangle = e^{iH_{0,S}\ t/\hbar}\vert\psi_S(t)\rangle$	$\vert\psi_S(t)\rangle = e^{-iH_S\ t/\hbar}\vert\psi_S(0)\rangle$
Observable	$A_H(t) = e^{iH_S\ t/\hbar}A_S e^{-iH_S\ t/\hbar}$	$A_I(t) = e^{iH_{0,S}\ t/\hbar}A_S e^{-iH_{0,S}\ t/\hbar}$	constant
Density matrix	constant	$\rho_I(t) = e^{iH_{0,S}\ t/\hbar}\rho_S(t)e^{-iH_{0,S}\ t/\hbar}$	$\rho_S(t) = e^{-iH_S\ t/\hbar}\rho_S(0)e^{iH_S\ t/\hbar}$

3 Different Answers

Pictures of Quantum Mechanics



Evolution	Picture		
of:	Heisenberg	Interaction	Schrödinger
Ket state	constant	$\lvert\psi_I(t)\rangle = e^{iH_{0,S}\ t/\hbar}\lvert\psi_S(t)\rangle$	$\lvert\psi_S(t)\rangle = e^{-iH_S\ t/\hbar}\lvert\psi_S(0)\rangle$
Observable	$A_H(t) = e^{iH_S\ t/\hbar}A_S e^{-iH_S\ t/\hbar}$	$A_I(t) = e^{iH_{0,S}\ t/\hbar}A_S e^{-iH_{0,S}\ t/\hbar}$	constant
Density matrix	constant	$\rho_I(t) = e^{iH_{0,S}\ t/\hbar}\rho_S(t)e^{-iH_{0,S}\ t/\hbar}$	$\rho_S(t) = e^{-iH_S\ t/\hbar}\rho_S(0)e^{iH_S\ t/\hbar}$

Table: https://i.stack.imgur.com/F6TZz.png

Measurement: Physical quantities

Classical mechanics: Time dependent variables (Vectors, Real Space)

Quantum mechanics: Operators, State functions (Linear Complex Vector Space)

Schrödinger Picture: Time-independent operators,
Time-dependent States

$$i\frac{\partial}{\partial t}\psi_S(\mathbf{x},t) = H^{op}\psi_S(\mathbf{x},t)$$

Expectation value: $\langle A(t)\rangle = \langle \psi_s(t)|O_s|\psi_s(t)\rangle$

Heisenberg Picture: Time dependent operators,
Time-independent States

$$i\frac{\partial}{\partial t}\psi_H(\mathbf{x},t) = 0$$

Time evolution operator $HU = i\frac{\partial}{\partial t}U$

Operators in Heisenberg Picture

$$\left\langle \hat{O}(t) \right\rangle = \left\langle \Psi_S^+(t) \middle| \hat{O}_S \middle| \Psi_S(t) \right\rangle$$

$$\Psi_S(t) \rightarrow \hat{U}(t,0)\Psi_H(0) \quad \Psi_S^+(t) \rightarrow \left(\hat{U}(t,0)\Psi_H(0)\right)^+ = \Psi_H^+(0)\hat{U}^+(t)$$

$$\left\langle \hat{O}(t) \right\rangle = \left\langle \Psi_H^+(0)\hat{U}^+(t,0) \middle| \hat{O}_S \middle| \hat{U}(t)\Psi_H(0) \right\rangle = \left\langle \Psi_H^+(0) \middle| \hat{U}^+(t)\hat{O}_S\hat{U}(t) \middle| \Psi_H(0) \right\rangle$$

$$= \left\langle \Psi_H^+(0) \middle| \hat{O}_H(t) \middle| \Psi_H(0) \right\rangle$$

$$\boxed{\hat{O}_H(t) = \hat{U}^+(t)\hat{O}_S\hat{U}(t) = e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}}$$

https://www.slideserve.com/avel/schr-dinger-heisenberg-interaction-pictures

Heisenberg EoM

$$\frac{\partial}{\partial t}\hat{O}_H(t) = \frac{\partial}{\partial t}\left(\hat{U}^+(t)\hat{O}_S\hat{U}(t)\right) = \frac{\partial}{\partial t}\hat{U}^+(t)\hat{O}_S\hat{U}(t) + \hat{U}^+(t)\hat{O}_S\frac{\partial}{\partial t}\hat{U}(t)$$

$$\frac{\partial}{\partial t}\hat{U} = -i\hat{H}\hat{U} \qquad \frac{\partial}{\partial t}\hat{U}^+ = +i\left(\hat{H}\hat{U}\right)^+ = +i\hat{U}^+\hat{H}$$

$$\begin{aligned}\frac{\partial}{\partial t}\hat{O}_H(t) &= i\left(\hat{U}^+\hat{H}\hat{O}_S\hat{U} - \hat{U}^+\hat{O}_S\hat{H}\hat{U}\right) = i\left(\hat{H}\hat{U}^+\hat{O}_S\hat{U} - \hat{U}^+\hat{O}_S\hat{U}\hat{H}\right) = i\left(\hat{H}\hat{O}_H - \hat{O}_H\hat{H}\right)\\ &= i\left[\hat{H},\hat{O}_H\right]\end{aligned}$$

$$\boxed{i\frac{\partial}{\partial t}\hat{O}_H(t) = \left[\hat{O}_H,\hat{H}\right]}$$

https://www.slideserve.com/avel/schr-dinger-heisenberg-interaction-pictures

Operators in Dirac Picture

$$\Psi_I(t) = e^{+i\hat{H}_o t}\Psi_S(t)$$

$$\Psi_S(t) = e^{-i(\hat{H}_1 + \hat{H}_o)t}\Psi_S(0)$$

EoM in Dirac Picture

$$\begin{aligned}
i\frac{\partial}{\partial t}\Psi_I(t) &= i(i\hat{H}_o)\Psi_I(t) + e^{+i\hat{H}_o t} i\frac{\partial}{\partial t}\Psi_S(t) \\
&= -\hat{H}_o\Psi_I(t) + e^{+i\hat{H}_o t}(\hat{H}_o + \hat{H}_1)\Psi_S(t) \\
&= -\hat{H}_o\Psi_I(t) + e^{+i\hat{H}_o t}(\hat{H}_o + \hat{H}_1)e^{-i\hat{H}_o t}\Psi_I(t) \\
&= e^{+i\hat{H}_o t}(-\hat{H}_o + \hat{H}_o + \hat{H}_1)e^{-i\hat{H}_o t}\Psi_I(t) \\
&= e^{+i\hat{H}_o t}\hat{H}_1 e^{-i\hat{H}_o t}\Psi_I(t)
\end{aligned}$$

https://www.slideserve.com/avel/schr-dinger-heisenberg-interaction-pictures

- Operators in interaction picture

$$i\frac{\partial}{\partial t}\Psi_I(t)=\hat{H}_1(t)\Psi_I(t) \qquad \hat{H}_1(t)=e^{+i\hat{H}_o t}\hat{H}_1 e^{-i\hat{H}_o t}$$

Recitation: Prove that $i\frac{\partial}{\partial t}\hat{O}_I(t)=\left[\hat{O}_I,\hat{H}_o\right]$

- Schrödinger picture

$$i\frac{\partial}{\partial t}\Psi_S(t)=\left(\hat{H}_o+\hat{H}_1\right)\Psi_S(t) \qquad \hat{O}_S=\hat{O}_S \qquad i\frac{\partial}{\partial t}\hat{O}_S=0$$

- Interaction picture

$$i\frac{\partial}{\partial t}\Psi_I(t)=\hat{H}_1(t)\Psi_I(t) \qquad \hat{O}_I(t)=e^{+i\hat{H}_o t}\hat{O}_S e^{-i\hat{H}_o t} \qquad i\frac{\partial}{\partial t}\hat{O}_I(t)=\left[\hat{O}_I,\hat{H}_o\right]$$

- Heisenberg picture

$$i\frac{\partial}{\partial t}\Psi_H(t)=0 \qquad \hat{O}_H(t)=e^{+i\hat{H}t}\hat{O}_S e^{-i\hat{H}t} \qquad i\frac{\partial}{\partial t}\hat{O}_H(t)=\left[\hat{O}_H,\hat{H}_H\right]$$

https://www.slideserve.com/avel/schr-dinger-heisenberg-interaction-pictures

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Week 13

Perturbation Theories

Ankara University | Physics Engineering Department

Dr. H. Ozgur Cildiroglu

Time-Independent Perturbation Theory

$$E_n|\psi_n\rangle = \hat{H}|\psi_n\rangle, \qquad E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$$

$$\hat{H} = \hat{H}^{(0)} + \lambda\hat{W}$$

$$\begin{aligned} E_n(\lambda) &= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots \\ &= \sum_{l=0}^{\infty} \lambda^l E_n^{(l)}, \\ |\psi_n(\lambda)\rangle &= |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \cdots \\ &= \sum_{l=0}^{\infty} \lambda^l |\psi_n^{(l)}\rangle. \end{aligned}$$

$$W_{mn} = \langle\psi_m^{(0)}|\hat{W}|\psi_n^{(0)}\rangle$$

$$\begin{aligned} &E_n^{(0)}|\psi_n^{(0)}\rangle + \\ &\lambda(E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle) + \\ &\lambda^2(E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle) + \\ &\cdots \\ &= \\ &\hat{H}^{(0)}|\psi_n^{(0)}\rangle + \\ &\lambda(\hat{H}^{(0)}|\psi_n^{(1)}\rangle + \hat{W}|\psi_n^{(0)}\rangle) + \\ &\lambda^2(\hat{H}^{(0)}|\psi_n^{(2)}\rangle + \hat{W}|\psi_n^{(1)}\rangle) + \\ &\cdots \end{aligned}$$

https://www.lancaster.ac.uk/staff/schomeru/lecturenotes/Quantum%20Mechanics/S11.html

- <u>0th Order Perturbation:</u> $\lambda^0 : E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$

- <u>1st Order Perturbation:</u>

- <u>2nd Order Perturbation:</u>

- Oth Order Perturbation:
- **1st Order Perturbation:**
- 2nd Order Perturbation:

$$\lambda^0 : \; E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$$

$$\lambda^1 : \quad E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(1)}\rangle + \hat{W}|\psi_n^{(0)}\rangle$$

$$E_n^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}\delta_{nm} = E_m^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle + W_{mn}.$$

$$\langle\psi_m^{(0)}|\hat{H}^{(0)}\psi_n^{(1)}\rangle = \langle\hat{H}^{(0)}\psi_m^{(0)}|\psi_n^{(1)}\rangle = E_m^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$$

$$E_n^{(1)} = W_{nn}$$

$$\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle = \frac{W_{mn}}{E_n^{(0)} - E_m^{(0)}}$$

$$|\psi_n^{(1)}\rangle = \sum_{m\neq n} \frac{W_{mn}}{E_n^{(0)} - E_m^{(0)}}|\psi_m^{(0)}\rangle$$

https://www.lancaster.ac.uk/staff/schomeru/lecturenotes/Quantum%20Mechanics/S11.html

- Oth Order Perturbation:

$$\lambda^0 : E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$$

- 1st Order Perturbation:

$$\lambda^1 : \quad E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(1)}\rangle + \hat{W}|\psi_n^{(0)}\rangle$$

- 2nd Order Perturbation:

- Recitation
  - .
  - .
  - .

$$E_n^{(2)} = \sum_{m \neq n} \frac{|W_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

Recitation: Harmonic Oscillator Problem

Obtain $\langle\psi_0^{(0)}|\hat{x}^2|\psi_0^{(0)}\rangle = \frac{\hbar}{2m\omega_0}$. (Givens $F = -m\omega_0^2 x + \lambda x$, $V = \frac{1}{2}m\omega_0^2 x^2 - \frac{1}{2}\lambda x^2$)

https://www.lancaster.ac.uk/staff/schomeru/lecturenotes/Quantum%20Mechanics/S11.html

Time-Dependent Perturbation Theories

$$\left.\begin{array}{l}
|\psi(t)\rangle_{\mathrm{I}} = e^{i\hat{H}_0 t/\hbar}|\psi(t)\rangle_{\mathrm{S}} \\[2ex]
i\hbar\partial_t|\psi(t)\rangle_{\mathrm{I}} = V_{\mathrm{I}}(t)|\psi(t)\rangle_{\mathrm{I}} \longrightarrow |\psi(t)\rangle_{\mathrm{I}} = \sum_n c_n(t)|n\rangle \\
\qquad\qquad\qquad\updownarrow \\
\qquad V_{\mathrm{I}}(t) = e^{i\hat{H}_0 t/\hbar} V e^{-i\hat{H}_0 t/\hbar}. \\[2ex]
\qquad i\hbar\dot{c}_m(t) = \displaystyle\sum_n V_{mn}(t) e^{i\omega_{mn}t} c_n(t) \\[2ex]
\qquad c_n(t) = c_n^{(0)} + c_n^{(1)}(t) + c_n^{(2)}(t) + \cdots
\end{array}\right\} \quad i\hbar\partial_t U_{\mathrm{I}}(t,t_0) = V_{\mathrm{I}}(t) U_{\mathrm{I}}(t,t_0)$$

$$U_{\mathrm{I}}(t,t_0) = \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} dt' V_{\mathrm{I}}(t') U_{\mathrm{I}}(t',t_0)$$

$$U_{\mathrm{I}}(t,t_0) = \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} dt' V_{\mathrm{I}}(t') + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' V_{\mathrm{I}}(t') \int_{t_0}^{t'} dt'' V_{\mathrm{I}}(t'') U_{\mathrm{I}}(t'',t_0)$$

$$U_{\mathrm{I}}(t,t_0) = \mathrm{T}\left[e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt' V_{\mathrm{I}}(t')}\right]$$

$$|i,t_0,t\rangle = U_{\mathrm{I}}(t,t_0)|i\rangle = \sum_n |n\rangle \overbrace{\langle n|U_{\mathrm{I}}(t,t_0)|i\rangle}^{c_n(t)} \quad \longleftarrow \quad \textstyle\sum_m |m\rangle\langle m| = \mathbb{I}$$

$$c_n(t) = \overbrace{\delta_{ni}}^{c_n^{(0)}} \overbrace{-\frac{i}{\hbar}\int_{t_0}^{t} dt' \langle n|V_{\mathrm{I}}(t')|i\rangle}^{c_n^{(1)}} \overbrace{-\frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \sum_m \langle n|V_{\mathrm{I}}(t')|m\rangle\langle m|V_{\mathrm{I}}(t'')|i\rangle}^{c_n^{(2)}} + \cdots$$

- Recall : $V_I = e^{i\hat{H}_0 t/\hbar} V e^{-i\hat{H}_0 t/\hbar}$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t} dt' e^{i\omega_{ni}t'} V_{ni}(t')$$

$$c_n^{(2)}(t) = -\frac{1}{\hbar^2}\sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{nm}t'+i\omega_{mi}t''} V_{nm}(t')V_{mi}(t'')$$

where $V_{nm}(t) = \langle n|V(t)|m\rangle$ and $\omega_{nm} = (E_n - E_m)/\hbar$,

with probabilities $P_{i\to n} = |c_n(t)|^2 = |c_n^{(1)}(t) + c_n^{(2)}(t) + \cdots|^2$

http://www.tcm.phy.cam.ac.uk/~bds10/aqp/handout_dep.pdf

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Additional Sources

A Brief Introduction to Relativistic Quantum Mechanics

Ankara University | Physics Engineering Department

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RQM

$$E = \frac{\boldsymbol{p}^2}{2m} + V,$$

$$E \rightarrow i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{p} \rightarrow -i\hbar\boldsymbol{\nabla},$$

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi.$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$$spin - 0$$

Klein-Gordon equation

$$spin - 1/2$$

Dirac equation

$$spin - 1$$

Proca equation

KG-Equation

$spin - 0$

$$E = \sqrt{p^2c^2 + m^2c^4}.$$

$$\sqrt{-\hbar^2c^2\nabla^2 + m^2c^2}\Psi = i\hbar\frac{\partial\Psi}{\partial t}.$$

$$\begin{aligned} E^2 &= p^2c^2 + m^2c^4 \\ \Rightarrow \left(i\hbar\frac{\partial}{\partial t}\right)^2\Psi &= -\hbar^2c^2\nabla^2 + m^2c^4\Psi \\ \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2\Psi - \nabla^2\Psi &\equiv \Box\Psi = -\frac{m^2c^2}{\hbar^2}\Psi, \end{aligned}$$

$$\Big\downarrow \quad \Box = \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2 = \partial_\mu\partial^\mu.$$

$$\Psi = \frac{1}{\sqrt{V}}\exp\left(\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}\right)\exp\left(-\frac{i}{\hbar}Et\right)$$

KG-Equation

$spin - 0$

$$E = \pm\sqrt{p^2c^2 + m^2c^4}.$$

!

Please also note that : $\exists$ *Negative energy solutions for each value of* $\boldsymbol{p}$.

For free particle system in positive state: $\nexists$ *transition mechanism to* , *negative energy state*.

If ∃ some external potentials, KG becomes

$$E \to E - e\phi, \quad \boldsymbol{p} \to \boldsymbol{p} - \frac{e}{c}\boldsymbol{A},$$

$$(i\hbar\partial_t - e\phi)^2\Psi = c^2(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A})^2\Psi + m^2c^4\Psi.$$

KG-Equation

$spin - 0$

$$-\frac{\hbar^2}{2m}(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*) \;=\; i\hbar(\Psi^*\dot{\Psi} + \Psi\dot{\Psi}^*)$$

➡ $$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*) \;=\; i\hbar\frac{\partial}{\partial t}(\Psi^*\Psi)$$

$$\rho_s \;=\; \Psi^*\Psi,\; \boldsymbol{j}_s \;=\; \tfrac{\hbar}{2mi}(\Psi^*\boldsymbol{\nabla}\Psi \;-\; \Psi\boldsymbol{\nabla}\Psi^*)$$

$$\frac{\partial\rho_s}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_s = 0$$

$$j^\mu = \alpha(\Psi\partial^\mu\Psi - \Psi\partial^\mu\Psi^*).$$

$$\Psi^*\Box\Psi \;=\; -\frac{m^2c^2}{\hbar}\Psi^*\Psi$$

$$\Psi\Box\Psi^* \;=\; -\frac{m^2c^2}{\hbar}\Psi\Psi^*$$

$$\partial_\mu j^\mu = 0,\; j^\mu = (j^0, \boldsymbol{j})$$

$$\alpha = -\tfrac{\hbar}{2mi}$$

$$\Psi^*\Box\Psi - \Psi\Box\Psi^* = \partial_\mu(\Psi^*\partial^\mu\Psi - \Psi\partial^\mu\Psi^*) = \rho = \frac{j^0}{c} = \frac{i\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

KG-Equation

$spin - 0$

$$-\frac{\hbar^2}{2m}(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*) \;=\; i\hbar(\Psi^*\dot{\Psi} + \Psi\dot{\Psi}^*)$$

$$\Longrightarrow \; -\frac{\hbar^2}{2m}\boldsymbol{\nabla}(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*) \;=\; i\hbar\frac{\partial}{\partial t}(\Psi^*\Psi)$$

$$\rho_s \;=\; \Psi^*\Psi,\; \boldsymbol{j}_s \;=\; \tfrac{\hbar}{2mi}(\Psi^*\boldsymbol{\nabla}\Psi \;-\; \Psi\boldsymbol{\nabla}\Psi^*)$$

$$j^\mu = \alpha(\Psi\partial^\mu\Psi - \Psi\partial^\mu\Psi^*),$$

$$\xrightarrow{\frac{\partial\rho_s}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_s = 0} \quad \partial_\mu j^\mu = 0,\; j^\mu = (j^0, \boldsymbol{j}) \qquad \alpha = -\tfrac{\hbar}{2mi}$$

$$\rho = \frac{j^0}{c} = \frac{i\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

$$\Longrightarrow \quad \rho_s = \Psi^*\Psi$$

Dirac Equation

$$spin-\frac{1}{2}$$

- For solving <0 prob. density problem of the KG-Eqn, we are searching for an eqn that consist first order derivative $\frac{d}{dt}$. It is hard to find sqrt of

$$-\hbar^2c^2\nabla^2+m^2c^4$$

for a single wave function. Consider ψ consists of N components ψ_l ;

$$\frac{1}{c}\frac{\partial\psi_l}{\partial t}+\sum_{k=1}^{3}\sum_{n=1}^{N}\alpha_{ln}^{k}\frac{\partial\psi_n}{\partial x^k}+\frac{imc}{\hbar}\sum_{n=1}^{N}\beta_{ln}\psi_n=0.$$

Dirac Equation

$$spin - \frac{1}{2}$$

$$\frac{1}{c}\frac{\partial \psi_l}{\partial t} + \sum_{k=1}^{3}\sum_{n=1}^{N} \alpha^k_{ln}\frac{\partial \psi_n}{\partial x^k} + \frac{imc}{\hbar}\sum_{n=1}^{N}\beta_{ln}\psi_n = 0.$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$$

α^k, β are $N \times N$ matrices.

$l = 1, 2, \ldots, N$, and $x^k = x,\ y,\ z,\ k = 1\,, 2,\ 3.$

Dirac Equation

$spin - \frac{1}{2}$

$$\frac{1}{c}\frac{\partial\psi}{\partial t} + \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \frac{imc}{\hbar}\beta\psi = 0$$

$$\big\downarrow$$

$$\boldsymbol{\alpha} = \alpha^1\hat{\boldsymbol{x}} + \alpha^2\hat{\boldsymbol{y}} + \alpha^3\hat{\boldsymbol{z}}.$$

$$\frac{1}{c}\left(\psi^\dagger\frac{\partial\psi}{\partial t} + \frac{\partial\psi^\dagger}{\partial t}\psi\right) + \boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\alpha}^\dagger\psi + \psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \frac{imc}{\hbar}(\psi^\dagger\beta\psi - \psi^\dagger\beta^\dagger\psi) = 0.$$

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$$

Dirac Equation

$$spin - \frac{1}{2}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot(\psi^\dagger\boldsymbol{\alpha}\psi) = 0$$

$$\boldsymbol{j} = c\psi^\dagger\boldsymbol{\alpha}\psi.$$

$$\left\{\ \text{if } \boldsymbol{\alpha}^\dagger = \boldsymbol{\alpha},\ \beta^\dagger = \beta.\right.$$

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} = \left(c\boldsymbol{\nabla}\cdot\frac{\hbar}{i}\boldsymbol{\nabla} + \beta mc^2\right)\psi.$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \frac{imc}{\hbar}\beta\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{imc}{\hbar}\beta\right)\psi = 0$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \alpha^i\alpha^j\partial_i\partial_j + \frac{m^2c^2}{\hbar^2}\beta^2 - \frac{imc}{\hbar}(\beta\alpha^i + \alpha^i\beta)\partial_i\right]\psi = 0$$

Dirac Equation

$$spin - \frac{1}{2}$$

$$\left.\begin{aligned} \alpha^i\alpha^j+\alpha^j\alpha^i &= 2\delta^{ij}I \\ \beta\alpha^i+\alpha^i\beta &= 0 \\ \beta^2 &= I \end{aligned}\right\} \quad \left.\begin{gathered} \beta\alpha^i = -\alpha^i\beta = (-I)\alpha^i\beta \\ \det\beta\det\alpha^i = (-1)^N\det\alpha^i\det\beta \end{gathered}\right\} \quad (\alpha^i)^{-1}\beta\alpha^i = -\beta$$

$$\mathrm{Tr}\left[(\alpha^i)^{-1}\beta\alpha^i\right] = \mathrm{Tr}\left[(\alpha^i\alpha^i)^{-1}\beta\right] = \mathrm{Tr}[\beta] = \mathrm{Tr}[-\beta]$$

$$\begin{aligned} \gamma^0 &= \beta, \\ \gamma^j &= \beta\alpha^j, \quad j=1,2,3 \\ \gamma^\mu &= (\gamma^0,\gamma^1,\gamma^2,\gamma^3), \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu \end{aligned}$$

$$\begin{aligned} i\beta\times\left(\frac{1}{c}\frac{\partial}{\partial t}+\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\frac{imc}{\hbar}\beta\right)\psi &= 0 \\ \Rightarrow\left(i\gamma^0\frac{\partial}{\partial x^0}+i\gamma^j\frac{\partial}{\partial x^j}-\frac{mc}{\hbar}\right)\psi &= \left(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar}\right)\psi = 0 \end{aligned}$$

Dirac Equation

$spin - \frac{1}{2}$

$$\gamma^\mu \partial_\mu \equiv \not{\partial},\ \gamma^\mu A_\mu \equiv \not{A}.$$

$$\left(i\ \not{\partial} - \frac{mc}{\hbar}\right)\psi = 0$$

$$\begin{aligned}
\gamma^{0\dagger} &= \gamma^0, \quad \text{(hermitian)} \\
\gamma^{j\dagger} &= (\beta\alpha^j)^\dagger = \alpha^{j\dagger}\beta^\dagger = \alpha^j\beta = -\beta\alpha^j = -\gamma^j, \\
\gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0, \\
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu}I. \quad \text{(Clifford algebra).}
\end{aligned}$$

$$\begin{aligned}
-i\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - \frac{mc}{\hbar}\psi^\dagger &= 0 \\
-i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 - \frac{mc}{\hbar}\psi^\dagger &= 0
\end{aligned}$$

$$\overline{\psi} \equiv \psi^\dagger\gamma^0$$

$$\boxed{i\partial_\mu\overline{\psi}\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} = 0}$$

$$\boxed{\frac{j^\mu}{c} = \overline{\psi}\gamma^\mu\psi = \left(\rho, \frac{\boldsymbol{j}}{c}\right), \quad \partial_\mu j^\mu = 0.}$$

https://cheng.physics.ucdavis.edu/teaching/230A-s08/rqm_all.pdf

Properties of Gamma Matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$$

$$\begin{aligned} (\gamma^0)^+ &= \gamma^0 \\ (\gamma^i)^+ &= -\gamma^i \end{aligned}$$

$$\left(\gamma^0\right)^2 = I$$

$$\left(\gamma^k\right)^2 = -I$$

$$\{\gamma^5, \gamma^k\} = 0$$

$$\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 0 \\ \gamma^\nu\gamma^\mu &= -\gamma^\mu\gamma^\nu \end{aligned}$$

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

$$\gamma^0 \;=\; \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\epsilon^{0123}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$$

$$\gamma^k \;=\; \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \longleftarrow \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = i\gamma^\mu\gamma^\nu$$

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\epsilon^{0123}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$$

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