Advanced Quantum Mechanics II

PEN425

Dr. H.Ozgur Cildiroglu

Advanced Quantum Mechanics II

PEN425

Week 9

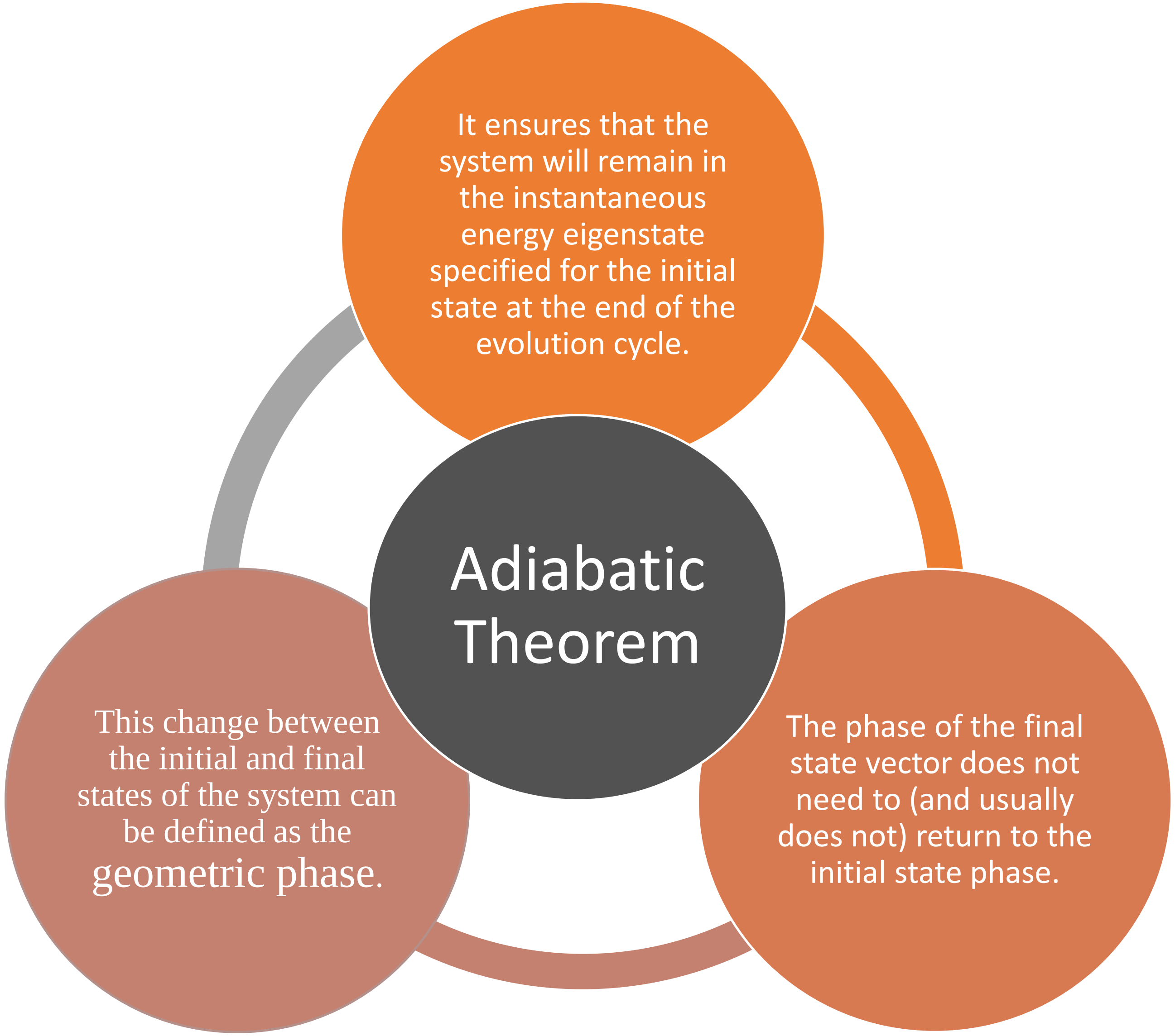
Adiabatic Theorem

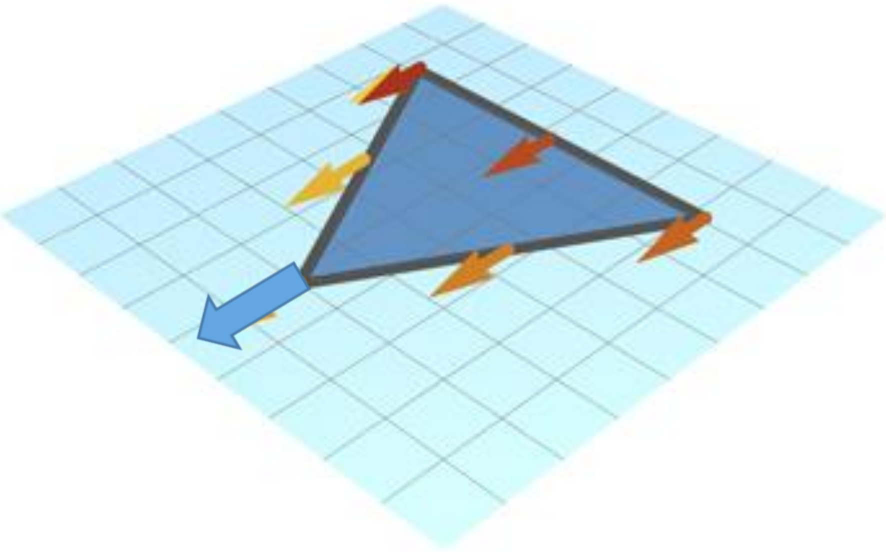
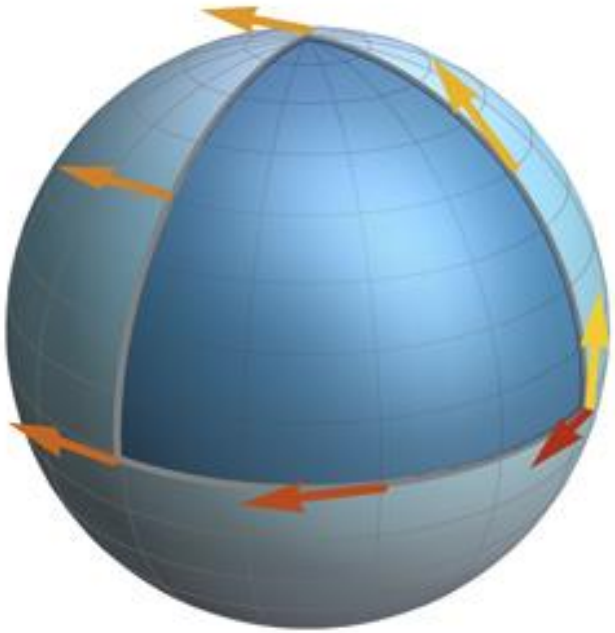
Quantum Mechanical Phases

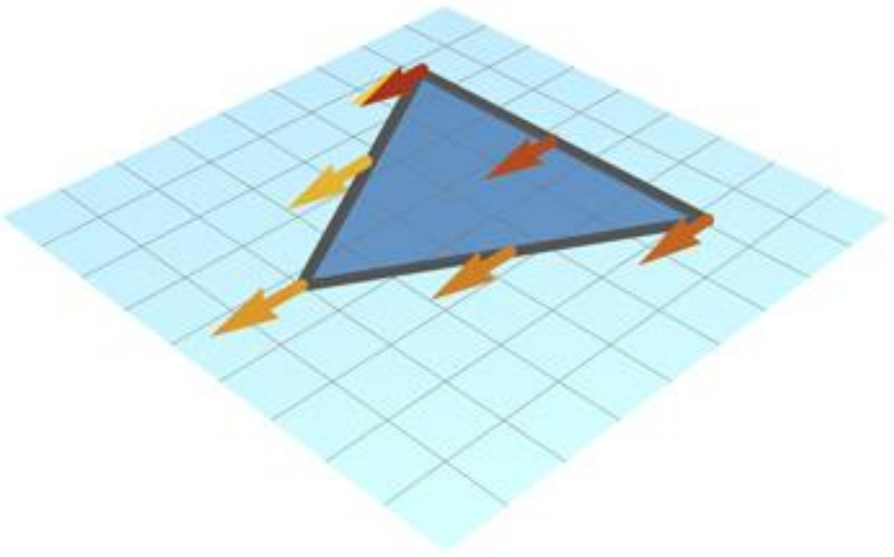
Geometric Berry Phase

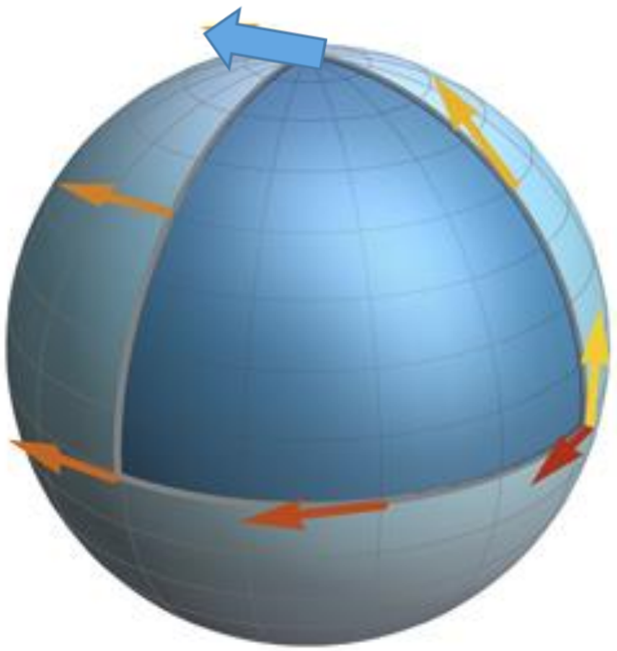
Ankara University | Physics Engineering Department

Dr. H. Ozgur Cildiroglu


It ensures that the system will remain in the instantaneous energy eigenstate specified for the initial state at the end of the evolution cycle.
Adiabatic Theorem
The phase of the final state vector does not need to (and usually does not) return to the initial state phase.
This change between the initial and final states of the system can be defined as the geometric phase.





Addition to the known dynamical phase →

$$i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle .$$

$$|\psi(t)\rangle = e^{i\phi_n}|n(t)\rangle$$

$$\theta_n(t) = -\int_0^t H_n(t')dt'$$

- Due to the adiabatic nature of the process, wavefunction of the system gains an additional phase known as the **geometric phase** in the literature.

- Total phase that the system will gain, including geometric phase γ_n,

$$\phi_n = \theta_n + \gamma_n$$

$$i\left[\frac{d}{dt}e^{i(\theta_n+\gamma_n)}\right]|n(t)\rangle + ie^{i(\theta_n+\gamma_n)}\frac{d}{dt}|n(t)\rangle = H(t)\,e^{i\phi_n}|n(t)\rangle$$

$$i(i\dot{\theta}_n + i\dot{\gamma}_n)|n(t)\rangle + i\frac{d}{dt}|n(t)\rangle = H(t)|n(t)\rangle$$

$$i\langle n(t)|\frac{d}{dt}|n(t)\rangle - \langle n(t)|(\dot{\theta}_n + \dot{\gamma}_n)|n(t)\rangle = \langle n(t)|H(t)|n(t)\rangle .$$

$$\begin{aligned} LHS &= -\int_0^t \left\langle n(t^{'})\right|\dot{\theta}_n\left|n(t^{'})\right\rangle dt^{'} - \int_0^t \left\langle n(t^{'})\right|\dot{\gamma}_n\left|n(t^{'})\right\rangle dt^{'} \\ &\quad + i\int_0^t \left\langle n(t^{'})\right|\frac{d}{dt^{'}}\left|n(t^{'})\right\rangle dt^{'} \end{aligned}$$

$$RHS = \int_0^t \left\langle n(t^{'})\right|H\left(t^{'}\right)\left|n(t^{'})\right\rangle dt^{'}$$

$$LHS = \int_0^t \left\langle n(t^{'})\right|\left[-H\left(t^{'}\right) + \frac{d}{dt^{'}}\int_0^t H\left(t^{'}\right)dt^{'}\right]\left|n(t^{'})\right\rangle dt^{'}$$

$$RHS = \int_0^t \left\langle n(t^{'})\right|\dot{\gamma}_n\left|n(t^{'})\right\rangle dt^{'} - i\int_0^t \left\langle n(t^{'})\right|\frac{d}{dt^{'}}\left|n(t^{'})\right\rangle dt^{'}$$

$$\gamma_n(t) - \gamma_n(0) = i\int_0^t \left\langle n(t^{'})\right|\frac{d}{dt^{'}}\left|n(t^{'})\right\rangle dt^{'}$$

$$\gamma_n = i\oint_C \left\langle n(t)\middle|\boldsymbol{\nabla} n(t)\right\rangle . d\mathbf{x}$$

$$
\begin{aligned}
0 &= \boldsymbol{\nabla} \left\langle n(t) \middle| n(t) \right\rangle \\
&= \left\langle \boldsymbol{\nabla} n(t) \middle| n(t) \right\rangle + \left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle \\
&= \left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle^{*} + \left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle \\
&= 2\mathfrak{Re} \left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle .
\end{aligned}
$$

$$
\left\langle n(t) \middle| \boldsymbol{\nabla} n(t) \right\rangle = \mathbf{A}_n
$$

$$
\oint_c \mathbf{F}.d\mathbf{l} = \iint_{s(c)} \boldsymbol{\nabla} \times \mathbf{F}.d\mathbf{S}
$$

$$\gamma_n = i \oint_c \mathbf{A_n}.d\mathbf{x} = i \iint_{s(c)} \mathbf{\nabla} \times \mathbf{A}.d\mathbf{S}.$$

$$\begin{aligned}\varepsilon_{ijk}\nabla^j A^k &= \varepsilon_{ijk}\nabla^j \left\langle n(t) \middle| \nabla^k n(t) \right\rangle \\ &= \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| \nabla^k n(t) \right\rangle + \varepsilon_{ijk} \left\langle n(t) \middle| \nabla^j \nabla^k n(t) \right\rangle\end{aligned}$$

$$\gamma_n = i \iint_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| \nabla^k n(t) \right\rangle dS_i$$

$$\sum_m \left| m \right\rangle \left\langle m \right| = I$$

$$\gamma_n = i \sum_m \iint_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle dS_i$$

$$\gamma_n = \iint\limits_{s(c)} \Big[i\varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| n \right\rangle \left\langle n \middle| \nabla^k n(t) \right\rangle$$

$$+ i \sum_{m \neq n} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle \Big] \, dS_i$$

$$= i \sum_{m \neq n} \iint\limits_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle dS_i$$

$$= -Im \sum_{m \neq n} \iint\limits_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle$$

$$\left\langle m \middle| n \right\rangle = \delta_{mn}$$

$$\begin{aligned} \boldsymbol{\nabla} \left\langle m \middle| n \right\rangle &= 0 \\ \left\langle \boldsymbol{\nabla} m \middle| n \right\rangle + \left\langle m \middle| \boldsymbol{\nabla} n \right\rangle &= 0 \\ \left\langle m \middle| \boldsymbol{\nabla} n \right\rangle &= -\left\langle n \middle| \boldsymbol{\nabla} m \right\rangle^* \end{aligned}$$

$$A_{mn}^{(i)} = \left\langle m \middle| \nabla^i n \right\rangle \quad \Longrightarrow \quad \gamma_n = i \sum_{m \neq n} \iint_{s(c)} \varepsilon_{ijk} A_{mn}^{(j)*} A_{mn}^{(k)} dS_i$$

$$\begin{aligned} \varepsilon_{ijk} A_{mn}^{(j)*} A_{mn}^{(k)} &= \varepsilon_{ijk} \left[R_{mn}^{(j)} - i \mathcal{J}_{mn}^{(j)} \right] \left[R_{mn}^{(k)} + i \mathcal{J}_{mn}^{(k)} \right] \\ &= \varepsilon_{ijk} \left[R_{mn}^{(j)} R_{mn}^{(k)} + \mathcal{J}_{mn}^{(j)} \mathcal{J}_{mn}^{(k)} \right] + i \varepsilon_{ijk} \left[R_{mn}^{(j)} \mathcal{J}_{mn}^{(k)} - \mathcal{J}_{mn}^{(j)} R_{mn}^{(k)} \right] \end{aligned}$$

$$\begin{aligned} \gamma_n &= -\sum_{m \neq n} \iint_{s(c)} \varepsilon_{ijk} \left[R_{mn}^{(j)} \mathcal{J}_{mn}^{(k)} - \mathcal{J}_{mn}^{(j)} R_{mn}^{(k)} \right] dS_i \\ &= -Im \sum_{m \neq n} \iint_{s(c)} \varepsilon_{ijk} \left\langle \nabla^j n(t) \middle| m \right\rangle \left\langle m \middle| \nabla^k n(t) \right\rangle dS_i \end{aligned}$$

$$[\nabla H(t)]\,|n(t)\rangle + H(t)\,|\nabla n(t)\rangle = [\nabla E_n(t)]\,|n(t)\rangle + E_n(t)\,|\nabla n(t)\rangle$$

$$\langle m|\nabla H(t)\,|n(t)\rangle + E_m(t)\,\langle m\,|\nabla n(t)\rangle = \nabla E_n(t)\,\langle m\,|n(t)\rangle + E_n(t)\,\langle m\,|\nabla n(t)\rangle$$

$$\langle m\,|\nabla n(t)\rangle = \frac{\langle m|\boldsymbol{\nabla} H(t)\,|n(t)\rangle}{E_n - E_m}$$

$$\gamma_n = -Im \sum_{m\neq n} \iint_{s(c)} \varepsilon_{ijk} \frac{\langle n(t)|\nabla^j H(t)\,|m\rangle\langle m|\nabla^k H(t)\,|n(t)\rangle}{(E_n - E_m)^2} dS_i.$$

Aharanov Anandan Phase

- Geometric Phase
- Projective Hilbert Space
- Adiabadic Approximation?
- Equivalance Classes
- Manifolds