# Advanced Quantum Mechanics II

PEN425

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Week 10

## Gauge Transformations

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# Gauge Transformations

- Classical Mechanics
- Electromagnetic Theory
- Quantum Mechanics

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- Quantum Mechanics

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

$$\boldsymbol{\nabla} \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

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$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla f(\mathbf{x}, t)$$

$$\phi \to \phi' = \phi - \frac{\partial f(\mathbf{x}, t)}{\partial t}$$

# Gauge Transformations in CM

$$\mathbf{F}_{Lorentz} = q\left[-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right]$$

$$\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \nabla(\mathbf{v}.\mathbf{A}) - (\mathbf{v}.\nabla)\,\mathbf{A}$$

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{A}$$

$$\mathbf{F}_{Lorentz} = q\left[-\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}(\mathbf{v}.\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right]$$

$$\mathbf{F}_{Lorentz} = -\frac{\partial}{\partial \mathbf{x}}\left[q\phi - q(\mathbf{v}.\mathbf{A})\right] + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\mathbf{x}}}\left(q\phi - q(\mathbf{v}.\mathbf{A})\right)\right]$$

# Gauge Transformations in CM

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$$U = q\left[\phi - (\mathbf{v}.\mathbf{A})\right]$$

$$\mathbf{F}_{Lorentz} = -\frac{\partial U}{\partial \mathbf{x}} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\mathbf{x}}}\right)$$

$$L = T - q\left[\phi + (\mathbf{v}.\mathbf{A})\right]$$

$$H = \frac{1}{2}m(\mathbf{p} - q\mathbf{A})^2 + q\phi.$$

$$\mathbf{p} \to \mathbf{p} - q\mathbf{A}$$

$$H \to H + q\phi$$

$$L^{'} = T - q\left[\phi^{'} - \left(\mathbf{v}.\mathbf{A}^{'}\right)\right]$$

$$L^{'} = T - q\left[\phi - \left(\mathbf{v}.\mathbf{A}\right)\right] + q\frac{\partial f}{\partial t} + q\mathbf{v}.\boldsymbol{\nabla} f$$

$$\frac{df}{dt} = \mathbf{v}.\nabla f + \frac{\partial f}{\partial t}$$

$$L^{'} = L + q\frac{df}{dt}$$

$$H^{'} = H - \frac{\partial F}{\partial t}$$

# In QM

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = H^{op}\psi(\mathbf{x},t)$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

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$$\begin{aligned} D_t &= \partial_t + ie\phi \\ \mathbf{D} &= \boldsymbol{\nabla} - ie\mathbf{A} \end{aligned}$$

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$$\begin{aligned} D_t\psi \rightarrow (D_t\psi)' &= D_t'\psi' &= U\left(D_t\psi\right) \\ \mathbf{D}\psi \rightarrow (\mathbf{D}\psi)' &= \mathbf{D}'\psi' &= U\left(\mathbf{D}\psi\right) \end{aligned}$$

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$$\left[iD_t^{'} + \frac{1}{2m}D'^2\right]\psi^{'} = 0.$$

$$\psi \to \psi^{'} = U\psi$$

$$\begin{aligned} D_t\psi \to (D_t\psi)^{'} &= D_t^{'}\psi^{'} &= U\left(D_t\psi\right) \\ \mathbf{D}\psi \to (\mathbf{D}\psi)^{'} &= \mathbf{D}^{'}\psi^{'} &= U\left(\mathbf{D}\psi\right) \end{aligned}$$

$$U = e^{ieg(\mathbf{x},t)}$$

$$D_t^{'} = \partial_t + ie\phi^{'}$$

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$$U = e^{ieg(\mathbf{x},t)}$$

$$\begin{aligned} D_t' &= \partial_t + ie\phi' \\ \mathbf{D}' &= \boldsymbol{\nabla} - ie\mathbf{A}' \end{aligned}$$

$$\left[\boldsymbol{\nabla} - ie\mathbf{A}'\right]U\psi = U\left[\boldsymbol{\nabla} - ie\mathbf{A}\right]\psi$$

$$U\boldsymbol{\nabla}\psi + \left(\boldsymbol{\nabla}U\right)\psi - ie\mathbf{A}'U\psi = U\boldsymbol{\nabla}\psi - ie\mathbf{A}U\psi$$

$$\left[\left(\mathbf{A}' - \mathbf{A}\right)U + \frac{i}{e}\boldsymbol{\nabla}U\right]\psi = 0$$

$$\mathbf{A}' = \mathbf{A} - \frac{i}{e}\left(\boldsymbol{\nabla}U\right)U^{-1}$$

$$\left(\boldsymbol{\nabla}U\right)U^{-1} = ie\boldsymbol{\nabla}g(\mathbf{x},t)$$

$$\boxed{\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}g(\mathbf{x},t).}$$

# Recitation

- Show that $\phi^{'} = \phi - \frac{\partial g}{\partial t}$ .