Advanced Quantum Mechanics II

PEN425

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Week 13

Perturbation Theories

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Time-Independent Perturbation Theory

$$E_n|\psi_n\rangle = \hat{H}|\psi_n\rangle, \qquad E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$$

$$\hat{H} = \hat{H}^{(0)} + \lambda\hat{W}$$

$$\begin{aligned} E_n(\lambda) &= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots \\ &= \sum_{l=0}^{\infty} \lambda^l E_n^{(l)}, \\ |\psi_n(\lambda)\rangle &= |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \cdots \\ &= \sum_{l=0}^{\infty} \lambda^l |\psi_n^{(l)}\rangle. \end{aligned}$$

$$W_{mn} = \langle\psi_m^{(0)}|\hat{W}|\psi_n^{(0)}\rangle$$

$$\begin{aligned} &E_n^{(0)}|\psi_n^{(0)}\rangle + \\ &\lambda(E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle) + \\ &\lambda^2(E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle) + \\ &\cdots \\ &= \\ &\hat{H}^{(0)}|\psi_n^{(0)}\rangle + \\ &\lambda(\hat{H}^{(0)}|\psi_n^{(1)}\rangle + \hat{W}|\psi_n^{(0)}\rangle) + \\ &\lambda^2(\hat{H}^{(0)}|\psi_n^{(2)}\rangle + \hat{W}|\psi_n^{(1)}\rangle) + \\ &\cdots \end{aligned}$$

https://www.lancaster.ac.uk/staff/schomeru/lecturenotes/Quantum%20Mechanics/S11.html

- <u>0th Order Perturbation:</u> $\lambda^0 : E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$

- <u>1st Order Perturbation:</u>

- <u>2nd Order Perturbation:</u>

- Oth Order Perturbation:
- **1st Order Perturbation:**
- 2nd Order Perturbation:

$$\lambda^0 : E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$$

$$\lambda^1 : \quad E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(1)}\rangle + \hat{W}|\psi_n^{(0)}\rangle$$

$$E_n^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}\delta_{nm} = E_m^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle + W_{mn}.$$

$$\langle\psi_m^{(0)}|\hat{H}^{(0)}\psi_n^{(1)}\rangle = \langle\hat{H}^{(0)}\psi_m^{(0)}|\psi_n^{(1)}\rangle = E_m^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$$

$$E_n^{(1)} = W_{nn}$$

$$\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle = \frac{W_{mn}}{E_n^{(0)} - E_m^{(0)}}$$

$$|\psi_n^{(1)}\rangle = \sum_{m\neq n} \frac{W_{mn}}{E_n^{(0)} - E_m^{(0)}}|\psi_m^{(0)}\rangle$$

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- Oth Order Perturbation:

$$\lambda^0 : E_n^{(0)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(0)}\rangle$$

- 1st Order Perturbation:

$$\lambda^1 : \quad E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle = \hat{H}^{(0)}|\psi_n^{(1)}\rangle + \hat{W}|\psi_n^{(0)}\rangle$$

- 2nd Order Perturbation:

- Recitation

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$$E_n^{(2)} = \sum_{m\neq n} \frac{|W_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

Recitation: Harmonic Oscillator Problem

Obtain $\langle\psi_0^{(0)}|\hat{x}^2|\psi_0^{(0)}\rangle = \frac{\hbar}{2m\omega_0}$. (Givens $F = -m\omega_0^2 x + \lambda x$, $V = \frac{1}{2}m\omega_0^2 x^2 - \frac{1}{2}\lambda x^2$)

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Time-Dependent Perturbation Theories

$$|\psi(t)\rangle_{\mathrm{I}} = e^{i\hat{H}_0 t/\hbar}|\psi(t)\rangle_{\mathrm{S}}$$

$$i\hbar\partial_t|\psi(t)\rangle_{\mathrm{I}} = V_{\mathrm{I}}(t)|\psi(t)\rangle_{\mathrm{I}} \longrightarrow |\psi(t)\rangle_{\mathrm{I}} = \sum_n c_n(t)|n\rangle$$

$$V_{\mathrm{I}}(t) = e^{i\hat{H}_0 t/\hbar} V e^{-i\hat{H}_0 t/\hbar}.$$

$$i\hbar\dot{c}_m(t) = \sum_n V_{mn}(t)e^{i\omega_{mn}t}c_n(t)$$

$$c_n(t) = c_n^{(0)} + c_n^{(1)}(t) + c_n^{(2)}(t) + \cdots$$

$$i\hbar\partial_t U_{\mathrm{I}}(t,t_0) = V_{\mathrm{I}}(t)U_{\mathrm{I}}(t,t_0)$$

$$U_{\mathrm{I}}(t,t_0) = \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} dt' V_{\mathrm{I}}(t') U_{\mathrm{I}}(t',t_0)$$

$$U_{\mathrm{I}}(t,t_0) = \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} dt' V_{\mathrm{I}}(t') + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' V_{\mathrm{I}}(t') \int_{t_0}^{t'} dt'' V_{\mathrm{I}}(t'') U_{\mathrm{I}}(t'',t_0)$$

$$U_{\mathrm{I}}(t,t_0) = \mathrm{T}\left[e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt' V_{\mathrm{I}}(t')}\right]$$

$$|i,t_0,t\rangle = U_{\mathrm{I}}(t,t_0)|i\rangle = \sum_n |n\rangle \overbrace{\langle n|U_{\mathrm{I}}(t,t_0)|i\rangle}^{c_n(t)}$$

$\sum_m |m\rangle\langle m| = \mathbb{I}$

$$c_n(t) = \underbrace{\delta_{ni}}_{c_n^{(0)}} \underbrace{-\frac{i}{\hbar}\int_{t_0}^{t} dt' \langle n|V_{\mathrm{I}}(t')|i\rangle}_{c_n^{(1)}} \underbrace{-\frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \sum_m \langle n|V_{\mathrm{I}}(t')|m\rangle\langle m|V_{\mathrm{I}}(t'')|i\rangle}_{c_n^{(2)}} + \cdots$$

- Recall : $V_{\mathrm{I}} = e^{i\hat{H}_0 t/\hbar} V e^{-i\hat{H}_0 t/\hbar}$

$$\boxed{\begin{aligned} c_n^{(1)}(t) &= -\frac{i}{\hbar}\int_{t_0}^{t} dt' e^{i\omega_{ni}t'} V_{ni}(t') \\ c_n^{(2)}(t) &= -\frac{1}{\hbar^2}\sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{nm}t' + i\omega_{mi}t''} V_{nm}(t') V_{mi}(t'') \end{aligned}}$$

where $V_{nm}(t) = \langle n|V(t)|m\rangle$ and $\omega_{nm} = (E_n - E_m)/\hbar$,

with probabilities $P_{i\to n} = |c_n(t)|^2 = |c_n^{(1)}(t) + c_n^{(2)}(t) + \cdots|^2$

http://www.tcm.phy.cam.ac.uk/~bds10/aqp/handout_dep.pdf