Advanced Quantum Mechanics II

PEN425

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Additional Sources

A Brief Introduction to Relativistic Quantum Mechanics

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RQM

$$E = \frac{\boldsymbol{p}^2}{2m} + V,$$

$$E \rightarrow i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{p} \rightarrow -i\hbar\boldsymbol{\nabla},$$

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi.$$

$$\left[i\partial_t + \frac{1}{2m}\nabla^2\right]\psi = 0$$

$spin - 0$

Klein-Gordon equation

$spin - 1/2$

Dirac equation

$spin - 1$

Proca equation

KG-Equation

$$spin - 0$$

$$E = \sqrt{p^2c^2 + m^2c^4}.$$

$$\sqrt{-\hbar^2c^2\nabla^2 + m^2c^2}\Psi = i\hbar\frac{\partial\Psi}{\partial t}.$$

$$\begin{aligned} E^2 &= p^2c^2 + m^2c^4 \\ \Rightarrow \left(i\hbar\frac{\partial}{\partial t}\right)^2\Psi &= -\hbar^2c^2\nabla^2 + m^2c^4\Psi \\ \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2\Psi - \nabla^2\Psi &\equiv \Box\Psi = -\frac{m^2c^2}{\hbar^2}\Psi, \end{aligned}$$

$$\Box = \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2 = \partial_\mu\partial^\mu.$$

↓

$$\Psi = \frac{1}{\sqrt{V}}\exp\left(\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}\right)\exp\left(-\frac{i}{\hbar}Et\right)$$

KG-Equation

$spin - 0$

$$E = \pm\sqrt{p^2c^2 + m^2c^4}.$$

Please also note that : $\exists$ $Negative\ energy\ solutions\ for\ each\ value\ of\ \boldsymbol{p}.$

For free particle system in positive state: $\nexists$ $transition\ mechanism\ to$, $negative\ energy\ state.$

If $\exists$ some external potentials, KG becomes

$$E \rightarrow E - e\phi, \quad \boldsymbol{p} \rightarrow \boldsymbol{p} - \frac{e}{c}\boldsymbol{A},$$

$$(i\hbar\partial_t - e\phi)^2\Psi = c^2(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A})^2\Psi + m^2c^4\Psi.$$

KG-Equation

$spin - 0$

$$-\frac{\hbar^2}{2m}(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*) = i\hbar(\Psi^*\dot{\Psi} + \Psi\dot{\Psi}^*)$$

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*) = i\hbar\frac{\partial}{\partial t}(\Psi^*\Psi)$$

$$\rho_s = \Psi^*\Psi, \; \boldsymbol{j}_s = \tfrac{\hbar}{2mi}(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*)$$

$$\frac{\partial\rho_s}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_s = 0$$

$$\Psi^*\Box\Psi = -\frac{m^2c^2}{\hbar}\Psi^*\Psi$$

$$\Psi\Box\Psi^* = -\frac{m^2c^2}{\hbar}\Psi\Psi^*$$

$$j^\mu = \alpha(\Psi\partial^\mu\Psi - \Psi\partial^\mu\Psi^*)$$

$$\partial_\mu j^\mu = 0, \; j^\mu = (j^0, \boldsymbol{j})$$

$$\alpha = -\tfrac{\hbar}{2mi}$$

$$\Psi^*\Box\Psi - \Psi\Box\Psi^* = \partial_\mu(\Psi^*\partial^\mu\Psi - \Psi\partial^\mu\Psi^*) = \rho = \frac{j^0}{c} = \frac{i\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

KG-Equation

$spin - 0$

$$-\frac{\hbar^2}{2m}(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*) = i\hbar(\Psi^*\dot{\Psi} + \Psi\dot{\Psi}^*)$$

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*) = i\hbar\frac{\partial}{\partial t}(\Psi^*\Psi)$$

$$\rho_s = \Psi^*\Psi, \; \boldsymbol{j}_s = \tfrac{\hbar}{2mi}(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*)$$

$$j^\mu = \alpha(\Psi\partial^\mu\Psi - \Psi\partial^\mu\Psi^*),$$

$$\frac{\partial\rho_s}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_s = 0$$

$$\partial_\mu j^\mu = 0, \; j^\mu = (j^0, \boldsymbol{j}) \qquad \alpha = -\tfrac{\hbar}{2mi}$$

$$\rho = \frac{j^0}{c} = \frac{i\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

$$\rho_s = \Psi^*\Psi$$

Dirac Equation

$$spin - \frac{1}{2}$$

- For solving <0 prob. density problem of the KG-Eqn, we are searching for an eqn that consist first order derivative $\frac{d}{dt}$. It is hard to find sqrt of

$$-\hbar^2 c^2 \nabla^2 + m^2 c^4$$

for a single wave function. Consider ψ consists of N components ψ_l ;

$$\frac{1}{c}\frac{\partial \psi_l}{\partial t} + \sum_{k=1}^{3}\sum_{n=1}^{N} \alpha_{ln}^{k} \frac{\partial \psi_n}{\partial x^k} + \frac{imc}{\hbar}\sum_{n=1}^{N} \beta_{ln}\psi_n = 0$$

Dirac Equation

$$spin - \frac{1}{2}$$

$$\frac{1}{c}\frac{\partial \psi_l}{\partial t} + \sum_{k=1}^{3}\sum_{n=1}^{N} \alpha^k_{ln}\frac{\partial \psi_n}{\partial x^k} + \frac{imc}{\hbar}\sum_{n=1}^{N}\beta_{ln}\psi_n = 0.$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$$

α^k, β are $N \times N$ matrices.

$l = 1, 2, \ldots, N$, and $x^k = x,\ y,\ z,\ \ k = 1, 2, 3.$

Dirac Equation

$$spin - \frac{1}{2}$$

$$\frac{1}{c}\frac{\partial \psi}{\partial t} + \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \psi + \frac{imc}{\hbar} \beta \psi = 0$$

$$\boldsymbol{\alpha} = \alpha^1 \hat{\boldsymbol{x}} + \alpha^2 \hat{\boldsymbol{y}} + \alpha^3 \hat{\boldsymbol{z}}.$$

$$\frac{1}{c}\left(\psi^\dagger \frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial t}\psi\right) + \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\alpha}^\dagger \psi + \psi^\dagger \boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\psi + \frac{imc}{\hbar}(\psi^\dagger \beta \psi - \psi^\dagger \beta^\dagger \psi) = 0.$$

$$\frac{\partial}{\partial t}(\psi^\dagger \psi) + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

Dirac Equation

$$spin-\frac{1}{2}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\psi^\dagger\psi)+\boldsymbol{\nabla}\cdot(\psi^\dagger\boldsymbol{\alpha}\psi)=0$$

$$\boldsymbol{j}=c\psi^\dagger\boldsymbol{\alpha}\psi.$$

if $\boldsymbol{\alpha}^\dagger=\boldsymbol{\alpha},\ \beta^\dagger=\beta.$

$$H\psi=i\hbar\frac{\partial\psi}{\partial t}=\left(c\boldsymbol{\nabla}\cdot\frac{\hbar}{i}\boldsymbol{\nabla}+\beta mc^2\right)\psi.$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t}-\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}-\frac{imc}{\hbar}\beta\right)\left(\frac{1}{c}\frac{\partial}{\partial t}+\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\frac{imc}{\hbar}\beta\right)\psi=0$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha^i\alpha^j\partial_i\partial_j+\frac{m^2c^2}{\hbar^2}\beta^2-\frac{imc}{\hbar}(\beta\alpha^i+\alpha^i\beta)\partial_i\right]\psi=0$$

Dirac Equation

$$spin - \frac{1}{2}$$

$$\left.\begin{aligned} \alpha^i\alpha^j + \alpha^j\alpha^i &= 2\delta^{ij} I \\ \beta\alpha^i + \alpha^i\beta &= 0 \\ \beta^2 &= I \end{aligned}\right\} \quad \left.\begin{aligned} \beta\alpha^i = -\alpha^i\beta = (-I)\alpha^i\beta \\ \det\beta\det\alpha^i = (-1)^N \det\alpha^i\det\beta \end{aligned}\right\} \quad (\alpha^i)^{-1}\beta\alpha^i = -\beta$$

$$\mathrm{Tr}\left[(\alpha^i)^{-1}\beta\alpha^i\right] = \mathrm{Tr}\left[(\alpha^i\alpha^i)^{-1}\beta\right] = \mathrm{Tr}[\beta] = \mathrm{Tr}[-\beta]$$

$$\begin{aligned} \gamma^0 &= \beta, \\ \gamma^j &= \beta\alpha^j, \quad j = 1, 2, 3 \\ \gamma^\mu &= (\gamma^0, \gamma^1, \gamma^2, \gamma^3), \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu \end{aligned}$$

$$\begin{aligned} i\beta \times \left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{imc}{\hbar}\beta\right)\psi &= 0 \\ \Rightarrow \left(i\gamma^0\frac{\partial}{\partial x^0} + i\gamma^j\frac{\partial}{\partial x^j} - \frac{mc}{\hbar}\right)\psi &= \left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 \end{aligned}$$

Dirac Equation

$$spin - \frac{1}{2}$$

$$\gamma^\mu \partial_\mu \equiv \not{\partial},\ \gamma^\mu A_\mu \equiv \not{A}.$$

$$\left(i\ \not{\partial} - \frac{mc}{\hbar}\right)\psi = 0$$

$$\begin{aligned}
\gamma^{0\dagger} &= \gamma^0, \quad \text{(hermitian)} \\
\gamma^{j\dagger} &= (\beta\alpha^j)^\dagger = \alpha^{j\dagger}\beta^\dagger = \alpha^j\beta = -\beta\alpha^j = -\gamma^j, \\
\gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0, \\
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu}I. \quad \text{(Clifford algebra)}.
\end{aligned}$$

$$\begin{aligned}
-i\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - \frac{mc}{\hbar}\psi^\dagger &= 0 \\
-i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 - \frac{mc}{\hbar}\psi^\dagger &= 0
\end{aligned}$$

$$\overline{\psi} \equiv \psi^\dagger\gamma^0$$

$$i\partial_\mu\overline{\psi}\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} = 0$$

$$\frac{j^\mu}{c} = \overline{\psi}\gamma^\mu\psi = \left(\rho, \frac{\boldsymbol{j}}{c}\right), \quad \partial_\mu j^\mu = 0.$$

https://cheng.physics.ucdavis.edu/teaching/230A-s08/rqm_all.pdf

Properties of Gamma Matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$$

$$(\gamma^0)^+ = \gamma^0$$
$$(\gamma^i)^+ = -\gamma^i$$

$$(\gamma^0)^2 = I$$
$$(\gamma^k)^2 = -I$$

$$\{\gamma^5, \gamma^k\} = 0$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$$
$$\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu$$

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = -i \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\epsilon^{0123}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \longleftarrow \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = i\gamma^\mu\gamma^\nu$$

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\epsilon^{0123}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$$

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