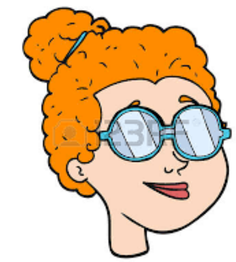
COUNTING I

Murat Osmanoglu

Counting

'soup' XOR 'sandwich'



'soup' AND 'sandwich'

- Ali and Buse eat lunch together in a specific restaurant regularly.

- restaurant has 9 choices for soup
  and 16 choices for sandwich

- How many different meals can Ali order ?

- How many different meals can Buse order ?

Counting

- Ali can order either one soup among 9 different soups or one sandwich among 16 different sandwiches.

- apply the Sum Rule here : 9 + 16 = 25 different choices

Sum Rule

- If A and B are disjoint finite sets, then the number of ways of choosing a single element from A or B is

$$|A \cup B| = |A| + |B|$$

- If a task can be done in one of n_1 ways or in one of n_2 ways such that none from n_1 ways is the same as any from n_2 ways, then there are $n_1 + n_2$ different ways to do the task

- If $A_1, A_2, \ldots, A_n$ are mutually disjoint sets ($A_i \cap A_j = \emptyset$), then the number of ways of choosing a single element from A_1 or A_2 or … A_n is

$$|A_1 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n|$$

Counting

- Buse can choose one one soup among 9 different choices; and for each choice of soup, she can have one sandwich among 16 different sandwiches. consider the meal as a pair (soup, sandwich)

- apply the Product Rule here : 9 . 16 = 144 different choices

Product Rule

- If A and B are finite sets, then the number of ways of choosing an element from A and an element from B is

$$|A \times B| = |A| \cdot |B|$$

- Suppose that a task can be broken into a sequence of two small tasks. If there are n_1 ways to do the first task, and for each one there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ different ways to finish the task

- If $A_1, A_2, \ldots, A_n$ are finite sets, then the number of ways of choosing an element from $A_1, \ldots$, an element from A_n is

$$|A_1 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

Counting

- How many bit-strings can you create with 3-digits ?

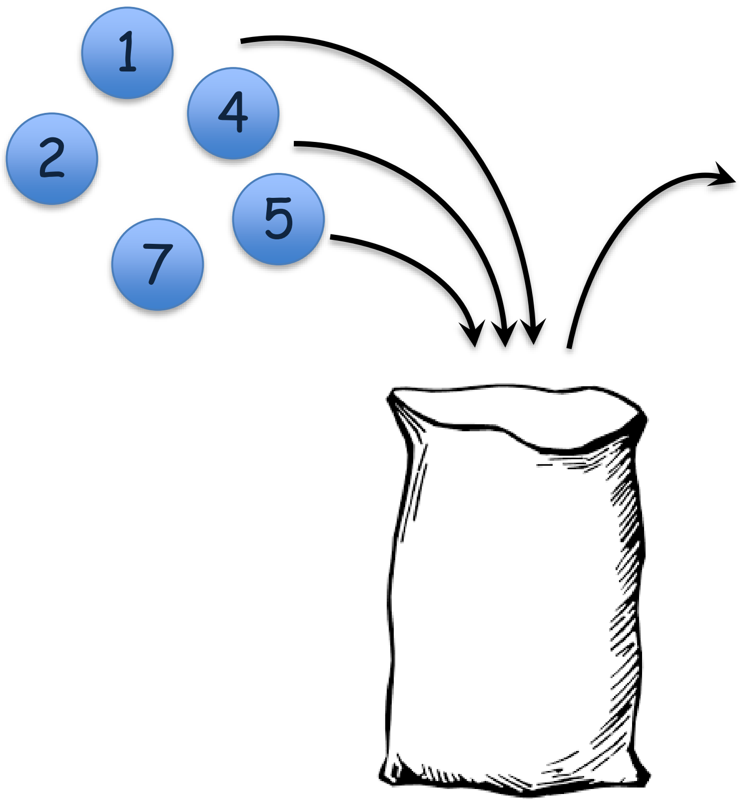
101, 001, 110, . . .

$$\underline{2} \times \underline{2} \times \underline{2} = 8$$

- What is the number of subsets of the set {a, b, c} ?

$$\{a, b, c\} \longrightarrow 1\ 1\ 1$$
$$\{a, b\} \longrightarrow 1\ 1\ 0$$
$$\{c\} \longrightarrow 0\ 0\ 1$$

Counting



you pick one ball at a time

How many 3-digits numbers can you create with the picked numbers ?

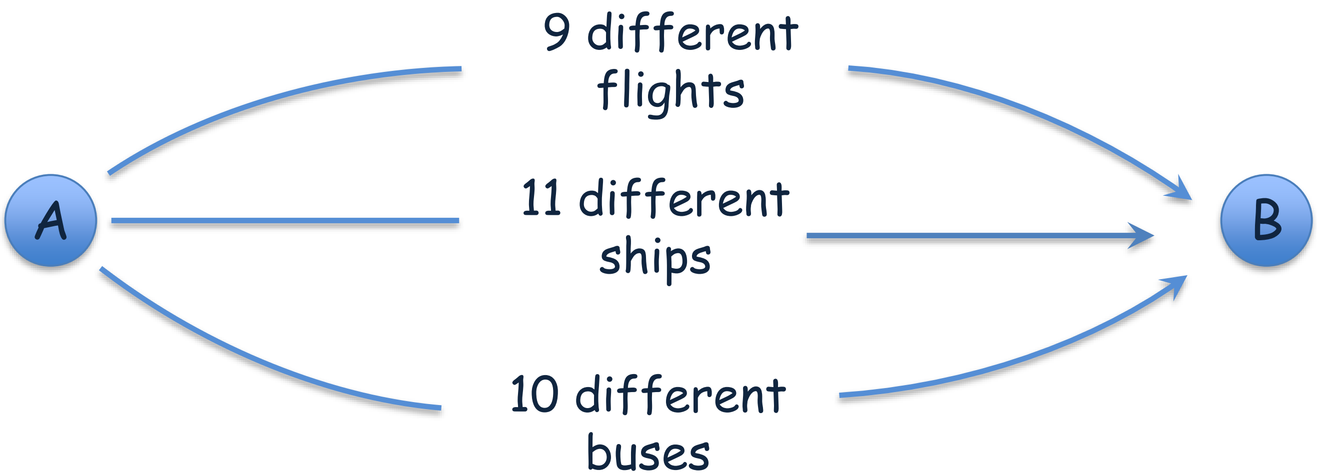
- If you leave them to the bag after you pick

$$\underline{\;5\;} \times \underline{\;5\;} \times \underline{\;5\;} = 125$$

- If you keep them after you pick

$$\underline{\;5\;} \times \underline{\;4\;} \times \underline{\;3\;} = 60$$

Counting



In how many different ways can you go from the city A to the city B ?

$$9 + 11 + 10 = 30$$

In how many different ways can you go to B and come back to A ?

$$30 \times 30 = 900$$

Counting

- You prepare a meal for your friends
- There are 5 kinds of bagels, 7 kinds of sandwiches, 6 drinks (hot coffee, hot tea, iced tea, cola, orange juice, apple juice)
- Bagels served with hot drinks and sandwiches served with cold drinks

How many different meals can you prepare?

(bagel, hot) or (sand, cold)

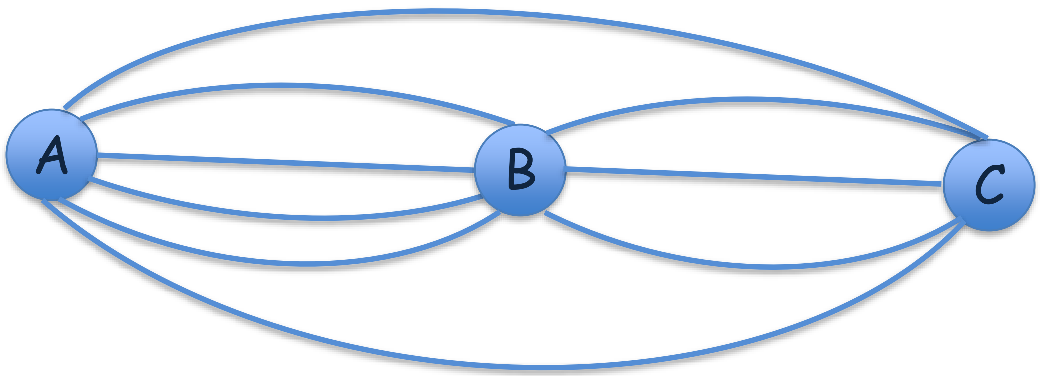
$$5 \times 2 \quad + \quad 7 \times 4 \quad = \quad 38$$

Counting

- How many different licence plates can be created if a plate contains of 2 digits indicating the city the car registered followed by 3 uppercase Turkish letters followed by 3 digits ?

$$\underbrace{\underline{81}}_{\text{indicating the city}} \times \underbrace{\underline{23} \times \underline{23} \times \underline{23}}_{\text{Turkish letters}} \times \underbrace{\underline{10} \times \underline{10} \times \underline{10}}_{\text{numbers}} = 985527000$$

Counting



In how many different ways can you go from the city A to the city C ?

$$4 \times 3 + 2 = 14$$

In how many different ways can you go to C and come back to A ?

$$14 \times 14 = 196$$

In how many different ways can you go to C and come back to A so that you can use same route to come back?

$$14 \times 13 = 182$$

Permutation

Assume there are 10 students. We choose five of them, and make them sit together to get a picture

How many such arrangements can we make?

- assign them numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- possible arrangements 13429, 60938, 19082, . . .

$$\underline{10}\times \underline{9}\times \underline{8}\times \underline{7}\times \underline{6} = 30240$$

How many different arrangements can we make for all students ?

$$\underline{10}\times \underline{9}\times \underline{8}\times \underline{7}\times \ldots \times \underline{1} = 3628800$$

Permutation

Assume there are 10 students. We choose five of them, and make them sit together to get a picture

How many such arrangements can we make?

- assign them numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- possible arrangements 13429, 60938, 19082, . . .

$$10 \times 9 \times 8 \times 7 \times 6 \; \frac{\times 5 \times 4 \times 3 \times 2 \times 1}{\times 5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{5!}$$

- the number of different permutation of size 5 for 10 objects
- the number of different permutation of size r for n objects

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutation

- Using the letters of the word 'COMPUTER' how many different words can you create ?

$$8!$$

- Using the letters of the word 'COMPUTER', how many different words of length 5 can you create ?

$$P(8, 5) = 8! / 5! = 336$$

- If repetitions are allowed, how many different words of length 5 can you create ?

$$8 \times 8 \times 8 \times 8 \times 8 = 32768$$

Permutation

- Using the letters of the word 'BALL', how many different words can you create?

$\cancel{4!}$ 12

BALL, BLAL, BLLA, ABLL, ALBL, ALLB
LBAL, LBLA, LABL, LALB, LLAB, LLBA

- Using the letters of the word 'ABARA', how many different words can you create?

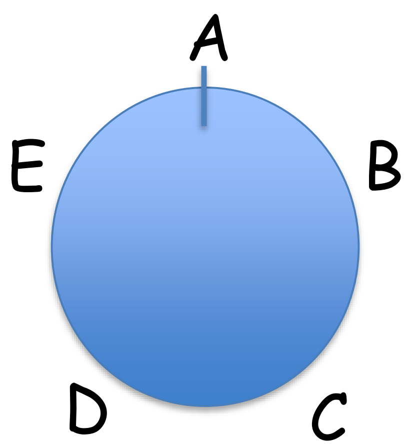
$A_1BA_3RA_2$ $A_2BA_3RA_1$ $A_3BA_2RA_1$
$A_1BA_2RA_3$ $A_2BA_1RA_3$ $A_3BA_1RA_2$

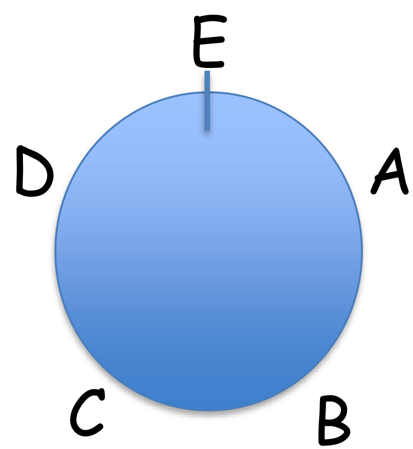
$5! / 3! = 20$

- pretend they are different A's
- fix other letters and reorder A's

Permutation

- There are 5 people : A, B, C, D, E
- They sit around a round table. How many different arrangements are possible ?





A B C D E
E A B C D
D E A B C
C D E A B
B C D E A

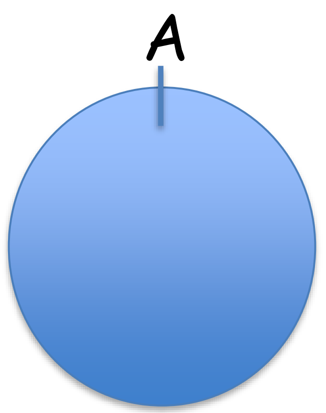
For each circular arrangement, there are 5 linear arrangements

5 x (# of circular) = (# of linear)

(# of circular) = 5! / 5 = 24

Permutation

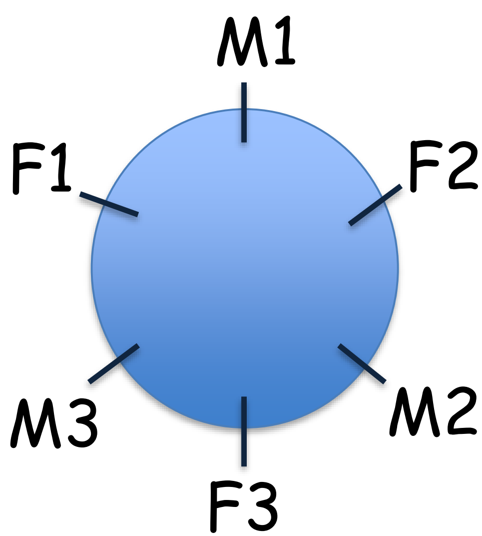
- There are 5 people : A, B, C, D, E
- They sit around a round table. How many different arrangements are possible ?



- fix one of them
- permute all others as in linear
- 4!

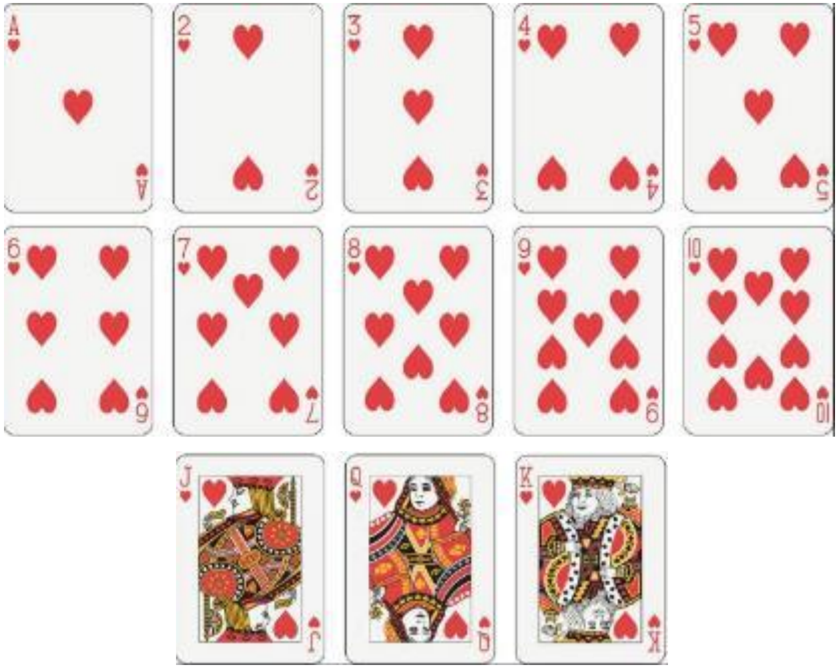
Permutation

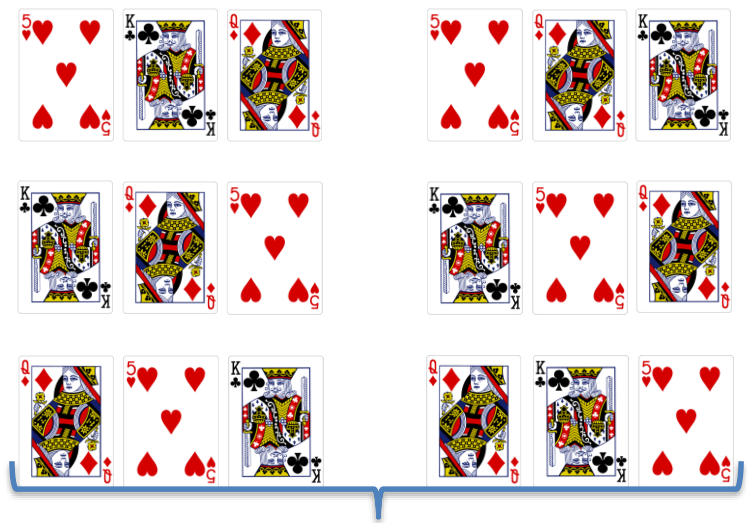
- You invite 2 couples for the dinner (3 couples at total)
- You have a around table. How many circular arrangements can you make such that no two women or no two men sit together ?



- $3 \times 2 \times 2 \times 1 \times 1 = 12$
- $3! \times 2! = 12$

Combinations





you count them as one

- Assume you play a game such that a player holds 3 cards.
- How many different hands can you create?

$$P(52,3) = 52! / (52-3)!$$

$$52! / [(52-3)! \,.\, 3!]$$

Combinations

The number of different selections of r elements out of n distinct objects :

$$C(n,r) = n! / [(n-r)!.r!]$$

There are 10 questions for an exam, and the student are required to choose 7 of them to answer.

- How many different answer sheets can a student prepare?

$$10! / (7! \,.\, 3!) = (10.9.8) / 3! = 120$$

- If the student picks 4 questions from the first 5 and 3 quesitions from the last 5, how many different answer sheets can he prepare?

1 2 3 4 5 6 7 8 9 10

(,)

$$\binom{5}{4} \text{x} \binom{5}{3} = \frac{5!}{1!.4!} \text{x} \frac{5!}{2!.3!} = 50$$

Combinations

The number of different selections of r elements out of n distinct objects :

$$C(n,r) = n! / [(n-r)!.r!]$$

There are 10 questions for an exam, and the student are required to choose 7 of them to answer.

- How many different answer sheets can a student prepare?

$$10! / (7! \,.\, 3!) = (10.9.8) / 3! = 120$$

- If the student is required to pick at least 3 questions from the first 5, how many different answer sheets can he prepare?

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2}$$

Combinations

- We are forming soccer teams with 6 players for a tournament from 30 boys and 25 girls. How many different 4 teams (2 for girls and 2 for boys) can you create ?

$$\binom{30}{6}\binom{24}{6}\binom{25}{6}\binom{19}{6}$$

- Suppose you are playing a game with five cards.

How many different hands can you have ? $\binom{52}{5}$

How many of them contains no club ? $\binom{39}{5}$

How many of them contains at least two clubs ?

$$\binom{13}{2}.\binom{13}{3}+\binom{13}{3}.\binom{13}{2}+\binom{13}{4}.\binom{13}{1}+\binom{13}{5}.\binom{13}{0}$$

Combinations

$\Sigma = \{0, 1\}$.
Let's use this alphabet to create three digits encoding:

000, 010, 111, 011, . . .

- How many encodings of length 8 can you create using the alphabet Σ ?

$$2 \times 2 \times \ldots \times 2 = 2^8$$

- How many of them contains exactly three 1's ?

$$\binom{8}{3}$$

__ 1 __ 1 1 __ __ __

__ __ 1 __ __ 1 1 __

Combinations

Σ = {0, 1}.
Let's use this alphabet to create three digits encoding:

000, 010, 111, 011, . . .

- How many encodings of length 8 can you create using the alphabet Σ ?

$$2 \times 2 \times \ldots \times 2 = 2^8$$

- How many of them contains exactly three 1's ?

$$\binom{8}{3}$$

Assume $x = x_1 x_2 \ldots x_n$, the weight of a given encoding is

$$w(x) = x_1 + x_2 + \ldots + x_n$$

- How many encodings of length n have even weight ?

$$\binom{8}{0} + \binom{8}{2} + \ldots + \binom{8}{8}$$

Combinations

$\Sigma = \{0, 1, 2\}$.
Let's use this alphabet to create three digits encoding:

020, 010, 112, 211, . . .

- How many encodings of length 8 can you create using the alphabet Σ ?

$$3 \times 3 \times \ldots \times 3 = 3^8$$

- How many of them contains exactly three 1's ?

$$\binom{8}{3} . 2^{8-3}$$

Assume $x = x_1 x_2 \ldots x_n$, the weight of a given encoding is

$$w(x) = x_1 + x_2 + \ldots + x_n$$

- How many encodings of length 8 have even weight ?

$$\binom{8}{0} 2^8 + \ldots + \binom{8}{8} 2^0$$

Binomial Theorem

- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
  $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Let x and y be variables and n be non-negative integer, then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \ldots + \binom{n}{n} y^n$$

- What is the coefficient of x^5y^2 in the expansion of $(x+y)^7$?

$$\binom{n}{i} x^{n-i}y^i \Rightarrow \mathrm{n} = 7 \text{ and } i = 2 \Rightarrow \binom{7}{2} x^5y^2 = \frac{7.6}{2} x^5y^2 = 21x^5y^2$$

- What is the coefficient of $x^{12}y^{15}$ in the expansion of $(2x-3y)^{25}$?

$$\binom{n}{i} x^{n-i}y^i \Rightarrow \mathrm{n} = 25 \text{ and } i = 15 \Rightarrow \binom{25}{15} (2x)^{10}(-3y)^{15}$$

$$\Rightarrow \binom{25}{15} 2^{10}(-3)^{15}x^{10}y^{15}$$

Binomial Theorem

Let x and y be variables and n be non-negative integer, then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n} y^n$$

- What is the coefficient of ${x_1}^{n_1} {x_2}^{n_2} \ldots {x_k}^{n_k}$ in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$?

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \ldots \binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1! \ldots n_k!}$$

- What is the coefficient of $x^3 y^2 z^2$ in the expansion of $(x+y+z)^7$?

$$\frac{7!}{3!\,2!\,2!} = 210$$

Binomial Theorem

- Prove that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

Suppose there is a set A={1, 2, . . ., n}. Let's write the elements of the power set P(A):

∅ $\binom{n}{0}$

{1}, {2}, {3}, . . ., {n} $\binom{n}{1}$

{1, 2}, {1, 3}, {1, 4}, . . ., {n-1, n} $\binom{n}{2}$

{1, 2, 3}, {1, 2, 4}, . . ., {n-2, n-1, n} $\binom{n}{3}$

. . .

{1, 2, . . ., n} $\binom{n}{n}$

|P(A)| = $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$

Binomial Theorem

- Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ where n is odd integer.

$$= (-1)^0\binom{n}{0} + (-1)^1\binom{n}{1} + (-1)^2\binom{n}{2} + (-1)^3\binom{n}{3} + \ldots + (-1)^n\binom{n}{n}$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + \binom{n}{n-1} - \binom{n}{n}$$

$$= \binom{n}{0} - \binom{n}{n} + \binom{n}{2} - \binom{n}{n-2} + \ldots + \binom{n}{n-1} - \binom{n}{1} = 0$$

- $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ and $\binom{n}{n-k} = \frac{n!}{k!(n-k)!}$

Thus, $\binom{n}{k} = \binom{n}{n-k}$

Binomial Theorem

- $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

There is a set A such that |A|=n+1
Assume one of them is a

A={ . . ., a, . . . }

$\{\ .\ .\ .\ .\ .\ .\ .\ \}$

create subsets of A with k elements

$\{\ a, \underbrace{.\ .\ .\ .\ .\ .}_{\text{k-1 elements}}\ \}$

$\{\ \underbrace{.\ .\ .\ .\ .\ .\ .}_{\text{k elements}}\ \}$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

Binomial Theorem

- $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Prove that $\sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2}$

$$= \binom{2}{2} + \binom{2}{1} + \binom{3}{1} + \ldots + \binom{n-1}{1} + \binom{n}{1}$$

$$\binom{1}{1} = \binom{2}{2}$$

Binomial Theorem

- $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Prove that $\sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2}$

$$= \binom{2}{2} + \binom{2}{1} + \binom{3}{1} + \ldots + \binom{n-1}{1} + \binom{n}{1}$$

$$= \binom{3}{2} + \binom{3}{1} + \ldots + \binom{n-1}{1} + \binom{n}{1}$$

$$= \binom{4}{2} + \ldots + \binom{n-1}{1} + \binom{n}{1}$$

$$= \binom{n}{2} + \binom{n}{1} = \binom{n+1}{2}$$